PBM

Problem Books in Mathematics

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The IMO Compendium

# The IMO Compendium

## A Collection of Problems Suggested for The International Mathematical Olympiads: 1959-2009

*Second Edition*

*2nd Ed.*

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## 3.46 The Forty-Sixth IMO
## Mérida, Mexico, July 8–19, 2005

### 3.46.1 Contest Problems

*First Day (July 13)*

1. Six points are chosen on the sides of an equilateral triangle $ABC$: $A_1, A_2$ on $BC$; $B_1, B_2$ on $CA$; $C_1, C_2$ on $AB$. These points are vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines $A_1B_2$, $B_1C_2$ and $C_1A_2$ are concurrent.
2. Let $a_1, a_2, \dots$ be a sequence of integers with infinitely many positive terms and infinitely many negative terms. Suppose that for each positive integer $n$, the numbers $a_1, a_2, \dots, a_n$ leave $n$ different remainders on division by $n$. Prove that each integer occurs exactly once in the sequence.
3. Let $x$, $y$, and $z$ be positive real numbers such that $xyz \geq 1$. Prove that
$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq 0.$$

*Second Day (July 14)*

4. Consider the sequence $a_1, a_2, \dots$ defined by
$$a_n = 2^n + 3^n + 6^n - 1 \quad (n = 1, 2, \dots).$$
Determine all positive integers that are relatively prime to every term of the sequence.
5. Let $ABCD$ be a given convex quadrilateral with sides $BC$ and $AD$ equal in length and not parallel. Let $E$ and $F$ be interior points of the sides $BC$ and $AD$ respectively such that $BE = DF$. The lines $AC$ and $BD$ meet at $P$; the lines $BD$ and $EF$ meet at $Q$; the lines $EF$ and $AC$ meet at $R$. Consider all the triangles $PQR$ as $E$ and $F$ vary. Show that the circumcircles of these triangles have a common point other than $P$.
6. In a mathematical competition, six problems were posed to the contestants. Each pair of problems was solved by more than $2/5$ of the contestants. Nobody solved all six problems. Show that there were at least two contestants who each solved exactly five problems.

### 3.46.2 Shortlisted Problems

1. **A1 (ROU)** Find all monic polynomials $p(x)$ with integer coefficients of degree two for which there exists a polynomial $q(x)$ with integer coefficients such that $p(x)q(x)$ is a polynomial having all coefficients $\pm 1$.

2. **A2 (BGR)** Let $\mathbb{R}^+$ denote the set of positive real numbers. Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that
$$f(x)f(y) = 2f(x+yf(x))$$
for all positive real numbers $x$ and $y$.
3. **A3 (CZE)** Four real numbers $p,q,r,s$ satisfy
$$p+q+r+s = 9 \quad \text{and} \quad p^2+q^2+r^2+s^2 = 21.$$
Prove that $ab-cd \geq 2$ holds for some permutation $(a,b,c,d)$ of $(p,q,r,s)$.
4. **A4 (IND)** Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation
$$f(x+y)+f(x)f(y) = f(xy)+2xy+1$$
for all real $x$ and $y$.
5. **A5 (KOR)**$^{\text{IMO3}}$ Let $x,y$ and $z$ be positive real numbers such that $xyz \geq 1$. Prove that
$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{y^5+z^2+x^2}+\frac{z^5-z^2}{z^5+x^2+y^2} \geq 0.$$
6. **C1 (AUS)** A house has an even number of lamps distributed among its rooms in such a way that there are at least three lamps in every room. Each lamp shares a switch with exactly one other lamp, not necessarily from the same room. Each change in the switch shared by two lamps changes their states simultaneously. Prove that for every initial state of the lamps there exists a sequence of changes in some of the switches at the end of which each room contains lamps that are on as well as lamps that are off.
7. **C2 (IRN)** Let $k$ be a fixed positive integer. A company has a special method to sell sombreros. Each customer can convince two persons to buy a sombrero after he/she buys one; convincing someone already convinced does not count. Each of these new customers can convince two others and so on. If each of the two customers convinced by someone makes at least $k$ persons buy sombreros (directly or indirectly), then that someone wins a free instructional video. Prove that if $n$ persons bought sombreros, then at most $n/(k+2)$ of them got videos.
8. **C3 (IRN)** In an $m \times n$ rectangular board of $mn$ unit squares, *adjacent* squares are ones with a common edge, and a *path* is a sequence of squares in which any two consecutive squares are adjacent. Each square of the board can be colored black or white. Let $N$ denote the number of colorings of the board such that there exists at least one black path from the left edge of the board to its right edge, and let $M$ denote the number of colorings in which there exist at least two nonintersecting black paths from the left edge to the right edge. Prove that $N^2 \geq 2^{mn}M$.
9. **C4 (COL)** Let $n \geq 3$ be a given positive integer. We wish to label each side and each diagonal of a regular $n$-gon $P_1 \dots P_n$ with a positive integer less than or equal to $r$ so that:

(i) every integer between 1 and $r$ occurs as a label;
(ii) in each triangle $P_iP_jP_k$ two of the labels are equal and greater than the third.
Given these conditions:
(a) Determine the largest positive integer $r$ for which this can be done.
(b) For that value of $r$, how many such labelings are there?

10. **C5 (SCG)** There are $n$ markers, each with one side white and the other side black, aligned in a row so that their white sides are up. In each step, if possible, we choose a marker with the white side up (but not one of the outermost markers), remove it, and reverse the closest marker to the left and the closest marker to the right of it. Prove that one can achieve the state with only two markers remaining if and only if $n-1$ is not divisible by 3.

11. **C6 (ROU)**IMO6 In a mathematical competition, six problems were posed to the contestants. Each pair of problems was solved by more than $2/5$ of the contestants. Nobody solved all six problems. Show that there were at least two contestants who each solved exactly five problems.

12. **C7 (USA)** Let $n \geq 1$ be a given integer, and let $a_1, \dots, a_n$ be a sequence of integers such that $n$ divides the sum $a_1 + \cdots + a_n$. Show that there exist permutations $\sigma$ and $\tau$ of $1, 2, \dots, n$ such that $\sigma(i) + \tau(i) \equiv a_i \pmod{n}$ for all $i = 1, \dots, n$.

13. **C8 (BGR)** Let $M$ be a convex $n$-gon, $n \geq 4$. Some $n-3$ of its diagonals are colored green and some other $n-3$ diagonals are colored red, so that no two diagonals of the same color meet inside $M$. Find the maximum possible number of intersection points of green and red diagonals inside $M$.

14. **G1 (HEL)** In a triangle $ABC$ satisfying $AB + BC = 3AC$ the incircle has center $I$ and touches the sides $AB$ and $BC$ at $D$ and $E$, respectively. Let $K$ and $L$ be the symmetric points of $D$ and $E$ with respect to $I$. Prove that the quadrilateral $ACKL$ is cyclic.

15. **G2 (ROU)**IMO1 Six points are chosen on the sides of an equilateral triangle $ABC$: $A_1, A_2$ on $BC$; $B_1, B_2$ on $CA$; $C_1, C_2$ on $AB$. These points are vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines $A_1B_2$, $B_1C_2$ and $C_1A_2$ are concurrent.

16. **G3 (UKR)** Let $ABCD$ be a parallelogram. A variable line $l$ passing through the point $A$ intersects the rays $BC$ and $DC$ at points $X$ and $Y$, respectively. Let $K$ and $L$ be the centers of the excircles of triangles $ABX$ and $ADY$, touching the sides $BX$ and $DY$, respectively. Prove that the size of angle $KCL$ does not depend on the choice of the line $l$.

17. **G4 (POL)**IMO5 Let $ABCD$ be a given convex quadrilateral with sides $BC$ and $AD$ equal in length and not parallel. Let $E$ and $F$ be interior points of the sides $BC$ and $AD$ respectively such that $BE = DF$. The lines $AC$ and $BD$ meet at $P$; the lines $BD$ and $EF$ meet at $Q$; the lines $EF$ and $AC$ meet at $R$. Consider all the triangles $PQR$ as $E$ and $F$ vary. Show that the circumcircles of these triangles have a common point other than $P$.

18. **G5 (ROU)** Let $ABC$ be an acute-angled triangle with $AB \neq AC$; let $H$ be its orthocenter and $M$ the midpoint of $BC$. Points $D$ on $AB$ and $E$ on $AC$ are such that $AE = AD$ and $D, H, E$ are collinear. Prove that $HM$ is orthogonal to the common chord of the circumcircles of triangles $ABC$ and $ADE$.

19. **G6 (RUS)** The median $AM$ of a triangle $ABC$ intersects its incircle $\omega$ at $K$ and $L$. The lines through $K$ and $L$ parallel to $BC$ intersect $\omega$ again at $X$ and $Y$. The lines $AX$ and $AY$ intersect $BC$ at $P$ and $Q$. Prove that $BP = CQ$.

20. **G7 (KOR)** In an acute triangle $ABC$, let $D, E, F, P, Q, R$ be the feet of perpendiculars from $A, B, C, A, B, C$ to $BC, CA, AB, EF, FD, DE$, respectively. Prove that $p(ABC)p(PQR) \geq p(DEF)^2$, where $p(T)$ denotes the perimeter of triangle $T$.

21. **N1 (POL)**[IMO4] Consider the sequence $a_1, a_2, \dots$ defined by

$$a_n = 2^n + 3^n + 6^n - 1 \quad (n = 1, 2, \dots).$$

Determine all positive integers that are relatively prime to every term of the sequence.

22. **N2 (NLD)**[IMO2] Let $a_1, a_2, \dots$ be a sequence of integers with infinitely many positive terms and infinitely many negative terms. Suppose that for each positive integer $n$, the numbers $a_1, a_2, \dots, a_n$ leave $n$ different remainders on division by $n$. Prove that each integer occurs exactly once in the sequence.

23. **N3 (MNG)** Let $a$, $b$, $c$, $d$, $e$, and $f$ be positive integers. Suppose that the sum $S = a+b+c+d+e+f$ divides both $abc+def$ and $ab+bc+ca-de-ef-fd$. Prove that $S$ is composite.

24. **N4 (COL)** Find all positive integers $n > 1$ for which there exists a unique integer $a$ with $0 < a \leq n!$ such that $a^n + 1$ is divisible by $n!$.

25. **N5 (NLD)** Denote by $d(n)$ the number of divisors of the positive integer $n$. A positive integer $n$ is called *highly divisible* if $d(n) > d(m)$ for all positive integers $m < n$. Two highly divisible integers $m$ and $n$ with $m < n$ are called consecutive if there exists no highly divisible integer $s$ satisfying $m < s < n$.
    (a) Show that there are only finitely many pairs of consecutive highly divisible integers of the form $(a, b)$ with $a \mid b$.
    (b) Show that for every prime number $p$ there exist infinitely many positive highly divisible integers $r$ such that $pr$ is also highly divisible.

26. **N6 (IRN)** Let $a$ and $b$ be positive integers such that $a^n + n$ divides $b^n + n$ for every positive integer $n$. Show that $a = b$.

27. **N7 (RUS)** Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, where $a_0, \dots, a_n$ are integers, $a_n > 0$, $n \geq 2$. Prove that there exists a positive integer $m$ such that $P(m!)$ is a composite number.

# 3.47 The Forty-Seventh IMO Ljubljana, Slovenia, July 6–18, 2006

## 3.47.1 Contest Problems

*First Day (July 12)*

1. Let $ABC$ be a triangle with incenter $I$. A point $P$ in the interior of the triangle satisfies
$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$
Show that $AP \geq AI$, and that equality holds if and only if $P = I$.
2. Let $\mathscr{P}$ be a regular 2006-gon. A diagonal of $\mathscr{P}$ is called *good* if its endpoints divide the boundary of $\mathscr{P}$ into two parts, each composed of an odd number of sides of $\mathscr{P}$. The sides of $\mathscr{P}$ are also called good.
Suppose $\mathscr{P}$ has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of $\mathscr{P}$. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.
3. Determine the least real number $M$ such that the inequality
$$\left|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)\right| \leq M(a^2+b^2+c^2)^2$$
holds for all real numbers $a$, $b$, and $c$.

*Second Day (July 13)*

4. Determine all pairs $(x,y)$ of integers such that
$$1+2^x+2^{2x+1}=y^2.$$
5. Let $P(x)$ be a polynomial of degree $n>1$ with integer coefficients and let $k$ be a positive integer. Consider the polynomial
$$Q(x)=P(P(\ldots P(P(x))\ldots)),$$
where $P$ occurs $k$ times. Prove that there are at most $n$ integers $t$ that satisfy the equality $Q(t)=t$.
6. Assign to each side $b$ of a convex polygon $\mathscr{P}$ the maximum area of a triangle that has $b$ as a side and is contained in $\mathscr{P}$. Show that the sum of the areas assigned to the sides of $\mathscr{P}$ is at least twice the area of $\mathscr{P}$.

## 3.47.2 Shortlisted Problems

1. **A1 (EST)** A sequence of real numbers $a_0, a_1, a_2, \ldots$ is defined by the formula
$$a_{i+1}=[a_i]\cdot\{a_i\}, \text{ for } i\geq 0;$$
here $a_0$ is an arbitrary number, $[a_i]$ denotes the greatest integer not exceeding $a_i$, and $\{a_i\}=a_i-[a_i]$. Prove that $a_i=a_{i+2}$ for $i$ sufficiently large.

2. **A2 (POL)** The sequence of real numbers $a_0, a_1, a_2, \dots$ is defined recursively by $a_0 = -1$ and
$$\sum_{k=0}^{n} \frac{a_{n-k}}{k+1} = 0, \quad \text{for } n \geq 1.$$
Show that $a_n > 0$ for $n \geq 1$.

3. **A3 (RUS)** The sequence $c_0, c_1, \dots, c_n, \dots$ is defined by $c_0 = 1$, $c_1 = 0$, and $c_{n+2} = c_{n+1} + c_n$ for $n \geq 0$. Consider the set $S$ of ordered pairs $(x, y)$ for which there is a finite set $J$ of positive integers such that $x = \sum_{j \in J} c_j$, $y = \sum_{j \in J} c_{j-1}$. Prove that there exist real numbers $\alpha, \beta$, and $M$ with the following property: an ordered pair of nonnegative integers $(x, y)$ satisfies the inequality $m < \alpha x + \beta y < M$ if and only if $(x, y) \in S$.
*Remark:* A sum over the elements of the empty set is assumed to be $0$.

4. **A4 (SRB)** Prove the inequality
$$\sum_{i<j} \frac{a_i a_j}{a_i + a_j} \leq \frac{n}{2(a_1 + a_2 + \cdots + a_n)} \sum_{i<j} a_i a_j$$
for positive real numbers $a_1, a_2, \dots, a_n$.

5. **A5 (KOR)** Let $a, b, c$ be the sides of a triangle. Prove that
$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} + \frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} + \frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 3.$$

6. **A6 (IRL)**[IMO3] Determine the smallest number $M$ such that the inequality
$$|ab(a^2-b^2) + bc(b^2-c^2) + ca(c^2-a^2)| \leq M(a^2+b^2+c^2)^2$$
holds for all real numbers $a, b, c$

7. **C1 (FRA)** We have $n \geq 2$ lamps $L_1, \dots, L_n$ in a row, each of them being either *on* or *off*. Every second we simultaneously modify the state of each lamp as follows: if the lamp $L_i$ and its neighbors (only one neighbor for $i = 1$ or $i = n$, two neighbors for other $i$) are in the same state, then $L_i$ is switched off; otherwise, $L_i$ is switched on.
Initially all the lamps are off except the leftmost one which is on.
   (a) Prove that there are infinitely many integers $n$ for which all the lamps will eventually be off.
   (b) Prove that there are infinitely many integers $n$ for which the lamps will never be all off.

8. **C2 (SRB)**[IMO2] A diagonal of a regular 2006-gon is called *odd* if its endpoints divide the boundary into two parts, each composed of an odd number of sides. Sides are also regarded as odd diagonals. Suppose the 2006-gon has been dissected into triangles by 2003 nonintersecting diagonals. Find the maximum possible number of isosceles triangles with two odd sides.

9. **C3 (COL)** Let $S$ be a finite set of points in the plane such that no three of them are on a line. For each convex polygon $P$ whose vertices are in $S$, let $a(P)$ be the number of vertices of $P$, and let $b(P)$ be the number of points of $S$ that are outside $P$. Prove that for every real number $x$

$$\sum_P x^{a(P)}(1-x)^{b(P)} = 1,$$

where the sum is taken over all convex polygons with vertices in $S$.
*Remark*. A line segment, a point, and the empty set are considered convex polygons of 2, 1, and 0 vertices respectively.

10. **C4 (TWN)** A cake has the form of an $n \times n$ square composed of $n^2$ unit squares. Strawberries lie on some of the unit squares so that each row and each column contains exactly one strawberry; call this arrangement $\mathscr{A}$.
Let $\mathscr{B}$ be another such arrangement. Suppose that every grid rectangle with one vertex at the top left corner of the cake contains no fewer strawberries of arrangement $\mathscr{B}$ than of arrangement $\mathscr{A}$. Prove that arrangement $\mathscr{B}$ can be obtained from $\mathscr{A}$ by performing a number of *switches*, defined as follows:
A *switch* consists in selecting a grid rectangle with only two strawberries, situated at its top right corner and bottom left corner, and moving these two strawberries to the other two corners of that rectangle.

11. **C5 (ARG)** An $(n,k)$-tournament is a contest with $n$ players held in $k$ rounds such that:
    (i) Each player plays in each round, and every two players meet at most once.
    (ii) If player $A$ meets player $B$ in round $i$, player $C$ meets player $D$ in round $i$, and player $A$ meets player $C$ in round $j$, then player $B$ meets player $D$ in round $j$.

    Determine all pairs $(n,k)$ for which there exists an $(n,k)$-tournament.

12. **C6 (COL)** A *holey triangle* is an upward equilateral triangle of side length $n$ with $n$ upward unit triangular holes cut out. A *diamond* is a $60°$–$120°$ unit rhombus. Prove that a holey triangle $T$ can be tiled with diamonds if and only if the following condition holds: every upward equilateral triangle of side length $k$ in $T$ contains at most $k$ holes, for $1 \le k \le n$.

13. **C7 (JPN)** Consider a convex polyhedron without parallel edges and without an edge parallel to any face other than the two faces adjacent to it. Call a pair of points of the polyhedron *antipodal* if there exist two parallel planes passing through these points and such that the polyhedron is contained between these planes.
Let $A$ be the number of antipodal pairs of vertices, and let $B$ be the number of antipodal pairs of midpoint edges. Determine the difference $A - B$ in terms of the numbers of vertices, edges, and faces.

14. **G1 (KOR)**[IMO1] Let $ABC$ be a triangle with incenter $I$. A point $P$ in the interior of the triangle satisfies $\angle PBA + \angle PCA = \angle PBC + \angle PCB$. Show that $AP \ge AI$ and that equality holds if and only if $P$ coincides with $I$.

15. **G2 (UKR)** Let $ABC$ be a trapezoid with parallel sides $AB > CD$. Points $K$ and $L$ lie on the line segments $AB$ and $CD$, respectively, so that $AK/KB = DL/LC$. Suppose that there are points $P$ and $Q$ on the line segment $KL$ satisfying $\angle APB = \angle BCD$ and $\angle CQD = \angle ABC$. Prove that the points $P, Q, B$, and $C$ are concyclic.

16. **G3 (USA)** Let $ABCDE$ be a convex pentagon such that $\angle BAC = \angle CAD = \angle DAE$ and $\angle ABC = \angle ACD = \angle ADE$. The diagonals $BD$ and $CE$ meet at $P$. Prove that the line $AP$ bisects the side $CD$.

17. **G4 (RUS)** A point $D$ is chosen on the side $AC$ of a triangle $ABC$ with $\angle C < \angle A < 90^\circ$ in such a way that $BD = BA$. The incircle of $ABC$ is tangent to $AB$ and $AC$ at points $K$ and $L$, respectively. Let $J$ be the incenter of triangle $BCD$. Prove that the line $KL$ intersects the line segment $AJ$ at its midpoint.

18. **G5 (HEL)** In triangle $ABC$, let $J$ be the center of the excircle tangent to side $BC$ at $A_1$ and to the extensions of sides $AC$ and $AB$ at $B_1$ and $C_1$, respectively. Suppose that the lines $A_1B_1$ and $AB$ are perpendicular and intersect at $D$. Let $E$ be the foot of the perpendicular from $C_1$ to line $DJ$. Determine the angles $\angle BEA_1$ and $\angle AEB_1$.

19. **G6 (BRA)** Circles $\omega_1$ and $\omega_2$ with centers $O_1$ and $O_2$ are externally tangent at point $D$ and internally tangent to a circle $\omega$ at points $E$ and $F$, respectively. Line $t$ is the common tangent of $\omega_1$ and $\omega_2$ at $D$. Let $AB$ be the diameter of $\omega$ perpendicular to $t$, so that $A$, $E$, and $O_1$ are on the same side of $t$. Prove that the lines $AO_1, BO_2, EF$, and $t$ are concurrent.

20. **G7 (SVK)** In a triangle $ABC$, let $M_a$, $M_b$, $M_c$, be respectively the midpoints of the sides $BC, CA, AB$, and let $T_a$, $T_b$, $T_c$ be the midpoints of the arcs $BC, CA, AB$ of the circumcircle of $ABC$, not counting the opposite vertices. For $i \in \{a,b,c\}$ let $\omega_i$ be the circle with $M_iT_i$ as diameter. Let $p_i$ be the common external tangent to $\omega_j$, $\omega_k$ ($\{i,j,k\} = \{a,b,c\}$) such that $\omega_i$ lies on the opposite side of $p_i$ from $\omega_j, \omega_k$. Prove that the lines $p_a$, $p_b$, $p_c$ form a triangle similar to $ABC$ and find the ratio of similitude.

21. **G8 (POL)** Let $ABCD$ be a convex quadrilateral. A circle passing through the points $A$ and $D$ and a circle passing through the points $B$ and $C$ are externally tangent at a point $P$ inside the quadrilateral. Suppose that $\angle PAB + \angle PDC \le 90^\circ$ and $\angle PBA + \angle PCD \le 90^\circ$. Prove that $AB + CD \ge BC + AD$.

22. **G9 (RUS)** Points $A_1$, $B_1$, $C_1$ are chosen on the sides $BC, CA, AB$ of a triangle $ABC$ respectively. The circumcircles of triangles $AB_1C_1, BC_1A_1, CA_1B_1$ intersect the circumcircle of triangle $ABC$ again at points $A_2, B_2, C_2$ respectively ($A_2 \ne A$, $B_2 \ne B$, $C_2 \ne C$). Points $A_3$, $B_3$, $C_3$ are symmetric to $A_1$, $B_1$, $C_1$ with respect to the midpoints of the sides $BC$, $CA$, $AB$, respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

23. **G10 (SRB)**[IMO6] Assign to each side $b$ of a convex polygon $\mathscr{P}$ the maximum area of a triangle that has $b$ as a side and is contained in $\mathscr{P}$. Show that the sum of the areas assigned to the sides of $\mathscr{P}$ is at least twice the area of $\mathscr{P}$.

24. **N1 (USA)**IMO4 Determine all pairs $(x,y)$ of integers satisfying the equation $1+2^x+2^{2x+1}=y^2$.

25. **N2 (CAN)** For $x\in(0,1)$ let $y\in(0,1)$ be the number whose $n$th digit after the decimal point is the $2^n$th digit after the decimal point of $x$. Show that if $x$ is rational then so is $y$.

26. **N3 (SAF)** The sequence $f(1), f(2), f(3), \dots$ is defined by

$$f(n)=\frac{1}{n}\left(\left[\frac{n}{1}\right]+\left[\frac{n}{2}\right]+\cdots+\left[\frac{n}{n}\right]\right),$$

where $[x]$ denotes the integral part of $x$.
(a) Prove that $f(n+1)>f(n)$ infinitely often.
(b) Prove that $f(n+1)<f(n)$ infinitely often.

27. **N4 (ROU)**IMO5 Let $P(x)$ be a polynomial of degree $n>1$ with integer coefficients and let $k$ be a positive integer. Consider the polynomial $Q(x)=P(P(\dots P(P(x))\dots))$, where $P$ occurs $k$ times. Prove that there are at most $n$ integers $t$ such that $Q(t)=t$.

28. **N5 (RUS)** Find all integer solutions of the equation

$$\frac{x^7-1}{x-1}=y^5-1.$$

29. **N6 (USA)** Let $a>b>1$ be relatively prime positive integers. Define the *weight* of an integer $c$, denoted by $w(c)$, to be the minimal possible value of $|x|+|y|$ taken over all pairs of integers $x$ and $y$ such that $ax+by=c$. An integer $c$ is called a *local champion* if $w(c)\geq w(c\pm a)$ and $w(c)\geq w(c\pm b)$. Find all local champions and determine their number.

30. **N7 (EST)** Prove that for every positive integer $n$ there exists an integer $m$ such that $2^m+m$ is divisible by $n$.

## 4.46 Solutions to the Shortlisted Problems of IMO 2005

1. Clearly, $p(x)$ has to be of the form $p(x) = x^2 + ax \pm 1$, where $a$ is an integer. For $a = \pm 1$ and $a = 0$, polynomial $p$ has the required property: it suffices to take $q = 1$ and $q = x + 1$, respectively.
   Suppose now that $|a| \geq 2$. Then $p(x)$ has two real roots, say $x_1, x_2$, which are also roots of $p(x)q(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, $a_i = \pm 1$. Thus
$$1 = \left| \frac{a_{n-1}}{x_i} + \cdots + \frac{a_0}{x_i^n} \right| \leq \frac{1}{|x_i|} + \cdots + \frac{1}{|x_i|^n} < \frac{1}{|x_i| - 1},$$
   which implies $|x_1|, |x_2| < 2$. This immediately rules out the case $|a| \geq 3$ and the polynomials $p(x) = x^2 \pm 2x - 1$. The remaining two polynomials $x^2 \pm 2x + 1$ satisfy the condition for $q(x) = x \mp 1$.
   Therefore, the polynomials $p(x)$ with the desired property are $x^2 \pm x \pm 1$, $x^2 \pm 1$, and $x^2 \pm 2x + 1$.

2. Given $y > 0$, consider the function $\varphi(x) = x + yf(x)$, $x > 0$. This function is injective: indeed, if $\varphi(x_1) = \varphi(x_2)$, then $f(x_1)f(y) = f(\varphi(x_1)) = f(\varphi(x_2)) = f(x_2)f(y)$, so $f(x_1) = f(x_2)$, so $x_1 = x_2$ by the definition of $\varphi$. Now if $x_1 > x_2$ and $f(x_1) < f(x_2)$, we have $\varphi(x_1) = \varphi(x_2)$ for $y = \frac{x_1 - x_2}{f(x_2) - f(x_1)} > 0$, which is impossible; hence $f$ is nondecreasing. The functional equation now yields $f(x)f(y) = 2f(x + yf(x)) \geq 2f(x)$ and consequently $f(y) \geq 2$ for $y > 0$. Therefore
$$f(x + yf(x)) = f(xy) = f(y + xf(y)) \geq f(2x)$$
   holds for arbitrarily small $y > 0$, implying that $f$ is constant on the interval $(x, 2x]$ for each $x > 0$. But then $f$ is constant on the union of all intervals $(x, 2x]$ over all $x > 0$, that is, on all of $\mathbb{R}^+$. Now the functional equation gives us $f(x) = 2$ for all $x$, which is clearly a solution.

   *Second Solution*. In the same way as above we prove that $f$ is nondecreasing, and hence its discontinuity set is at most countable. We can extend $f$ to $\mathbb{R} \cup \{0\}$ by defining $f(0) = \inf_x f(x) = \lim_{x \to 0} f(x)$, and the new function $f$ is continuous at 0 as well. If $x$ is a point of continuity of $f$ we have $f(x)f(0) = \lim_{y \to 0} f(x)f(y) = \lim_{y \to 0} 2f(x + yf(x)) = 2f(x)$, hence $f(0) = 2$. Now, if $f$ is continuous at $2y$, then $2f(y) = \lim_{x \to 0} f(x)f(y) = \lim_{x \to 0} 2f(x + yf(x)) = 2f(2y)$. Thus $f(y) = f(2y)$, for all but countably many values of $y$. Being nondecreasing $f$ is a constant; hence $f(x) = 2$.

3. Assume without loss of generality that $p \geq q \geq r \geq s$. We have
$$(pq + rs) + (pr + qs) + (ps + qr) = \frac{(p + q + r + s)^2 - p^2 - q^2 - r^2 - s^2}{2} = 30.$$
   It is easy to see that $pq + rs \geq pr + qs \geq ps + qr$, which gives us $pq + rs \geq 10$. Now setting $p + q = x$, we obtain $x^2 + (9 - x)^2 = (p + q)^2 + (r + s)^2 = 21 + 2(pq + rs) \geq 41$, which is equivalent to $(x - 4)(x - 5) \geq 0$. Since $x = p + q \geq r + s$, we conclude that $x \geq 5$. Thus

$$25 \le p^2+q^2+2pq = 21-(r^2+s^2)+2pq \le 21+2(pq-rs),$$

or $pq-rs \ge 2$, as desired.
*Remark*. The quadruple $(p,q,r,s)=(3,2,2,2)$ shows that the estimate 2 is the best possible.

4. Setting $y=0$ yields $(f(0)+1)(f(x)-1)=0$, and since $f(x)=1$ for all $x$ is impossible, we get $f(0)=-1$. Now plugging in $x=1$ and $y=-1$ gives us $f(1)=1$ or $f(-1)=0$. In the first case setting $x=1$ in the functional equation yields $f(y+1)=2y+1$, i.e., $f(x)=2x-1$, which is one solution.
Suppose now that $f(1)=a\neq 1$ and $f(-1)=0$. Plugging $(x,y)=(z,1)$ and $(x,y)=(-z,-1)$ in the functional equation yields

$$\begin{aligned} f(z+1) &= (1-a)f(z)+2z+1 \\ f(-z-1) &= f(z)+2z+1. \end{aligned}$$

It follows that $f(z+1)=(1-a)f(-z-1)+a(2z+1)$, i.e. $f(x)=(1-a)f(-x)+a(2x-1)$. Analogously, $f(-x)=(1-a)f(x)+a(-2x-1)$, which together with the previous equation yields

$$(a^2-2a)f(x) = -2a^2x-(a^2-2a).$$

Now $a=2$ is clearly impossible. For $a\notin\{0,2\}$ we get $f(x)=\frac{-2ax}{a-2}-1$. This function satisfies the requirements only for $a=-2$, giving the solution $f(x)=-x-1$. In the remaining case, when $a=0$, we have $f(x)=f(-x)$. Setting $y=z$ and $y=-z$ in the functional equation and subtracting yields $f(2z)=4z^2-1$, so $f(x)=x^2-1$, which satisfies the equation.
Thus the solutions are $f(x)=2x-1$, $f(x)=-x-1$, and $f(x)=x^2-1$.

5. The desired inequality is equivalent to

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2}+\frac{x^2+y^2+z^2}{y^5+z^2+x^2}+\frac{x^2+y^2+z^2}{z^5+x^2+y^2} \le 3. \tag{1}$$

By the Cauchy inequality we have $(x^5+y^2+z^2)(yz+y^2+z^2) \ge (x^{5/2}(yz)^{1/2}+y^2+z^2)^2 \ge (x^2+y^2+z^2)^2$ and therefore

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2} \le \frac{yz+y^2+z^2}{x^2+y^2+z^2}.$$

We get analogous inequalities for the other two summands in (1). Summing these yields

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2}+\frac{x^2+y^2+z^2}{y^5+z^2+x^2}+\frac{x^2+y^2+z^2}{z^5+x^2+y^2} \le 2+\frac{xy+yz+zx}{x^2+y^2+z^2},$$

which together with the well-known inequality $x^2+y^2+z^2 \ge xy+yz+zx$ gives us the result.

*Second solution*. Multiplying both sides by the common denominator and using notation in Chapter 2 (*Muirhead's inequality*), we get

$$T_{5,5,5}+4T_{7,5,0}+T_{5,2,2}+T_{9,0,0}\geq T_{5,5,2}+T_{6,0,0}+2T_{5,4,0}+2T_{4,2,0}+T_{2,2,2}.$$

By Schur's and Muirhead's inequalities we have that $T_{9,0,0}+T_{5,2,2}\geq 2T_{7,2,0}\geq 2T_{7,1,1}$. Since $xyz\geq 1$ we have that $T_{7,1,1}\geq T_{6,0,0}$. Therefore

$$T_{9,0,0}+T_{5,2,2}\geq 2T_{6,0,0}\geq T_{6,0,0}+T_{4,2,0}. \tag{2}$$

Moreover, Muirhead's inequality combined with $xyz\geq 1$ gives us $T_{7,5,0}\geq T_{5,5,2}$, $2T_{7,5,0}\geq 2T_{6,5,1}\geq 2T_{5,4,0}$, $T_{7,5,0}\geq T_{6,4,2}\geq T_{4,2,0}$, and $T_{5,5,5}\geq T_{2,2,2}$. Adding these four inequalities to (2) yields the desired result.

6. A room will be called *economic* if some of its lamps are on and some are off. Two lamps sharing a switch will be called *twins*. The twin of a lamp $l$ will be denoted by $\bar{l}$.
Suppose we have arrived at a state with the minimum possible number of uneconomic rooms, and that this number is strictly positive. Let us choose any uneconomic room, say $R_0$, and a lamp $l_0$ in it. Let $\bar{l}_0$ be in a room $R_1$. Switching $l_0$, we make $R_0$ economic; therefore, since the number of uneconomic rooms cannot be decreased, this change must make room $R_1$ uneconomic. Now choose a lamp $l_1$ in $R_1$ having the twin $\bar{l}_1$ in a room $R_2$. Switching $l_1$ makes $R_1$ economic, and thus must make $R_2$ uneconomic. Continuing in this manner we obtain a sequence $l_0, l_1, \dots$ of lamps with $l_i$ in a room $R_i$ and $\bar{l}_i\neq l_{i+1}$ in $R_{i+1}$ for all $i$. The lamps $l_0, l_1, \dots$ are switched in this order. This sequence has the property that switching $l_i$ and $\bar{l}_i$ makes room $R_i$ economic and room $R_{i+1}$ uneconomic.
Let $R_m = R_k$ with $m > k$ be the first repetition in the sequence $(R_i)$. Let us stop switching the lamps at $l_{m-1}$. The room $R_k$ was uneconomic prior to switching $l_k$. Thereafter, lamps $l_k$ and $\bar{l}_{m-1}$ have been switched in $R_k$, but since these two lamps are distinct (indeed, their twins $\bar{l}_k$ and $l_{m-1}$ are distinct), the room $R_k$ is now economic, as well as all the rooms $R_0$, $R_1$, $\dots$, $R_{m-1}$. This decreases the number of uneconomic rooms, contradicting our assumption.

7. Let $v$ be the number of video winners. One easily finds that for $v=1$ and $v=2$, the number $n$ of customers is at least $2k+3$ and $3k+5$ respectively. We prove by induction on $v$ that if $n\geq k+1$, then $n\geq (k+2)(v+1)-1$.
Without loss of generality, we can assume that the total number $n$ of customers is minimum possible for given $v>0$. Consider a person $P$ who was convinced by nobody but himself. Then $P$ must have won a video; otherwise, $P$ could be removed from the group without decreasing the number of video winners. Let $Q$ and $R$ be the two persons convinced by $P$. We denote by $\mathscr{C}$ the set of persons influenced by $P$ through $Q$ to buy a sombrero, including $Q$, and by $\mathscr{D}$ the set of all other customers excluding $P$. Let $x$ be the number of video winners in $\mathscr{C}$. Then there are $v-x-1$ video winners in $\mathscr{D}$. We have $|\mathscr{C}|\geq (k+2)(x+1)-1$, by the induction hypothesis if $x>0$ and because $P$ is a winner if $x=0$. Similarly, $|\mathscr{D}|\geq (k+2)(v-x)-1$. Thus $n\geq 1+(k+2)(x+1)-1+(k+2)(v-x)-1$, i.e., $n\geq (k+2)(v+1)-1$.

8. Suppose that a two-sided $m \times n$ board $T$ is considered, where exactly $k$ of the squares are transparent. A transparent square is colored only on one side (then it looks the same from the other side), while a nontransparent one needs to be colored on both sides, not necessarily in the same color.
Let $C = C(T)$ be the set of colorings of the board in which there exist two black paths from the left edge to the right edge, one on top and one underneath, not intersecting at any transparent square. If $k = 0$ then $|C| = N^2$. We prove by induction on $k$ that $2^k|C| \le N^2$. This will imply the statement of the problem, since $|C| = M$ for $k = mn$.
Let $q$ be a fixed transparent square. Consider any coloring $B$ in $C$: If $q$ is converted into a nontransparent square, a new board $T'$ with $k-1$ transparent squares is obtained, so by the induction hypothesis $2^{k-1}|C(T')| \le N^2$. Since $B$ contains two black paths at most one of which passes through $q$, coloring $q$ in either color on the other side will result in a coloring in $C'$; hence $|C(T')| \ge 2|C(T)|$, implying $2^k|C(T)| \le N^2$ and finishing the induction.
*Second solution.* By a *path* we shall mean a black path from the left edge to the right edge. Let $\mathscr{A}$ denote the set of pairs of $m \times n$ boards each of which has a path. Let $\mathscr{B}$ denote the set of pairs of boards such that the first board has two nonintersecting paths. Obviously, $|\mathscr{A}| = N^2$ and $|\mathscr{B}| = 2^{mn}M$. To prove $|\mathscr{A}| \ge |\mathscr{B}|$, we will construct an injection $f : \mathscr{B} \to \mathscr{A}$.
Among paths on a given board we define path $x$ to be *lower* than $y$ if the set of squares “under” $x$ is a subset of the squares under $y$. This relation is a relation of incomplete order. However, for each board with at least one path there exists a lowest path (comparing two intersecting paths, we can always take the “lower branch” on each nonintersecting segment). Now, for a given element of $\mathscr{B}$, we “swap” the lowest path and all squares underneath on the first board with the corresponding points on the other board. This swapping operation is the desired injection $f$. Indeed, since the first board still contains the highest path (which didn’t intersect the lowest one), the new configuration belongs to $\mathscr{A}$. On the other hand, this configuration uniquely determines the lowest path on the original element of $\mathscr{B}$; hence no two different elements of $\mathscr{B}$ can go to the same element of $\mathscr{A}$. This completes the proof.

9. Let $[XY]$ denote the label of segment $XY$, where $X$ and $Y$ are vertices of the polygon. Consider any segment $MN$ with the maximum label $[MN] = r$. By condition (ii), for any $P_i \ne M,N$, exactly one of $P_iM$ and $P_iN$ is labeled by $r$. Thus the set of all vertices of the $n$-gon splits into two complementary groups: $\mathscr{A} = \{P_i \mid [P_iM] = r\}$ and $\mathscr{B} = \{P_i \mid [P_iN] = r\}$. We claim that a segment $XY$ is labelled by $r$ if and only if it joins two points from different groups. Assume without loss of generality that $X \in \mathscr{A}$. If $Y \in \mathscr{A}$, then $[XM] = [YM] = r$, so $[XY] < r$. If $Y \in \mathscr{B}$, then $[XM] = r$ and $[YM] < r$, so $[XY] = r$ by (ii), as we claimed.
We conclude that a labeling satisfying (ii) is uniquely determined by groups $\mathscr{A}$ and $\mathscr{B}$ and labelings satisfying (ii) within $A$ and $B$.

(a) We prove by induction on $n$ that the greatest possible value of $r$ is $n-1$. The degenerate cases $n=1,2$ are trivial. If $n \geq 3$, the number of different labels of segments joining vertices in $\mathscr{A}$ (resp. $\mathscr{B}$) does not exceed $|\mathscr{A}|-1$ (resp. $|\mathscr{B}|-1$), while all segments joining a vertex in $\mathscr{A}$ and a vertex in $\mathscr{B}$ are labeled by $r$. Therefore $r \leq (|\mathscr{A}|-1)+(|\mathscr{B}|-1)+1=n-1$. Equality is achieved if all the mentioned labels are different.

(b) Let $a_n$ be the number of labelings with $r=n-1$. We prove by induction that $a_n = \frac{n!(n-1)!}{2^{n-1}}$. This is trivial for $n=1$, so let $n \geq 2$. If $|\mathscr{A}|=k$ is fixed, the groups $\mathscr{A}$ and $\mathscr{B}$ can be chosen in $\binom{n}{k}$ ways. The set of labels used within $\mathscr{A}$ can be selected among $1,2,\dots,n-2$ in $\binom{n-2}{k-1}$ ways. Now the segments within groups $\mathscr{A}$ and $\mathscr{B}$ can be labeled so as to satisfy (ii) in $a_k$ and $a_{n-k}$ ways, respectively. In this way, every labeling has been counted twice, since choosing $\mathscr{A}$ is equivalent to choosing $\mathscr{B}$. It follows that

$$\begin{aligned} a_n &= \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}\binom{n-2}{k-1}a_k a_{n-k} \\ &= \frac{n!(n-1)!}{2(n-1)}\sum_{k=1}^{n-1}\frac{a_k}{k!(k-1)!}\cdot\frac{a_{n-k}}{(n-k)!(n-k-1)!} \\ &= \frac{n!(n-1)!}{2(n-1)}\sum_{k=1}^{n-1}\frac{1}{2^{k-1}}\cdot\frac{1}{2^{n-k-1}} = \frac{n!(n-1)!}{2^{n-1}}. \end{aligned}$$

10. Denote by $L$ the leftmost and by $R$ the rightmost marker. To start with, note that the parity of the number of black-side-up markers remains unchanged. Hence, if only two markers remain, these markers must have the same color up.

We shall show by induction on $n$ that the game can be successfully finished if and only if $n \equiv 0$ or $n \equiv 2 \pmod 3$, and that the upper sides of $L$ and $R$ will be black in the first case and white in the second case.

The statement is clear for $n=2,3$. Assume that we have finished the game for some $n$, and denote by $k$ the position of the marker $X$ (counting from the left) that was last removed. Having finished the game, we have also finished the subgames with the $k$ markers from $L$ to $X$ and with the $n-k+1$ markers from $X$ to $R$ (inclusive). Thereby, before $X$ was removed, the upper side of $L$ had been black if $k \equiv 0$ and white if $k \equiv 2 \pmod 3$, while the upper side of $R$ had been black if $n-k+1 \equiv 0$ and white if $n-k+1 \equiv 2 \pmod 3$. Markers $L$ and $R$ were reversed upon the removal of $X$. Therefore, in the final position, $L$ and $R$ are white if and only if $k \equiv n-k+1 \equiv 0$, which yields $n \equiv 2 \pmod 3$, and black if and only if $k \equiv n-k+1 \equiv 2$, which yields $n \equiv 0 \pmod 3$.

On the other hand, a game with $n$ markers can be reduced to a game with $n-3$ markers by removing the second, fourth, and third markers in this order. This finishes the induction.

*Second solution.* An invariant can be defined as follows. To each white marker with $k$ black markers to its left we assign the number $(-1)^k$. Let $S$ be the sum of the assigned numbers. Then it is easy to verify that the remainder of $S$ modulo

3 remains unchanged throughout the game: For example, when a white marker with two white neighbors and $k$ black markers to its left is removed, $S$ decreases by $3(-1)^t$.

Initially, $S = n$. In the final position with two markers remaining, $S$ equals 0 if the two markers are black and 2 if these are white (note that, as before, the two markers must be of the same color). Thus $n \equiv 0$ or $2 \pmod 3$.

Conversely, a game with $n$ markers is reduced to $n-3$ markers as in the first solution.

11. Assume that there were $n$ contestants, $a_i$ of whom solved exactly $i$ problems, where $a_0 + \cdots + a_5 = n$. Let us count the number $N$ of pairs $(C,P)$, where contestant $C$ solved the pair of problems $P$. Each of the 15 pairs of problems was solved by at least $\frac{2n+1}{5}$ contestants, implying $N \geq 15 \cdot \frac{2n+1}{5} = 6n+3$. On the other hand, $a_i$ students solved $\frac{i(i-1)}{2}$ pairs; hence

$$6n+3 \leq N \leq a_2 + 3a_3 + 6a_4 + 10a_5 = 6n + 4a_5 - (3a_3 + 5a_2 + 6a_1 + 6a_0).$$

Consequently $a_5 \geq 1$. Assume that $a_5 = 1$. Then we must have $N = 6n+4$, which is possible only if 14 of the pairs of problems were solved by exactly $\frac{2n+1}{5}$ students and the remaining one by $\frac{2n+1}{5}+1$ students, and all students but the winner solved 4 problems.

The problem $t$ not solved by the winner will be called *tough* and the pair of problems solved by $\frac{2n+1}{5}+1$ students *special*.

Let us count the number $M_p$ of pairs $(C,P)$ for which $P$ contains a fixed problem $p$. Let $b_p$ be the number of contestants who solved $p$. Then $M_t = 3b_t$ (each of the $b_t$ students solved three pairs of problems containing $t$), and $M_p = 3b_p + 1$ for $p \neq t$ (the winner solved four such pairs). On the other hand, each of the five pairs containing $p$ was solved by $\frac{2n+1}{5}$ or $\frac{2n+1}{5}+1$ students, so $M_p = 2n+2$ if the special pair contains $p$, and $M_p = 2n+1$ otherwise.

Now since $M_t = 3b_t = 2n+1$ or $2n+2$, we have $2n+1 \equiv 0$ or $2 \pmod 3$. But if $p \neq t$ is a problem not contained in the special pair, we have $M_p = 3b_p + 1 = 2n+1$; hence $2n+1 \equiv 1 \pmod 3$, which is a contradiction.

12. Suppose that there exist desired permutations $\sigma$ and $\tau$ for some sequence $a_1, \dots, a_n$. Given a sequence $(b_i)$ with sum divisible by $n$ that differs modulo $n$ from $(a_i)$ in only two positions, say $i_1$ and $i_2$, we show how to construct desired permutations $\sigma'$ and $\tau'$ for sequence $(b_i)$. In this way, starting from an arbitrary sequence $(a_i)$ for which $\sigma$ and $\tau$ exist, we can construct desired permutations for any other sequence with sum divisible by $n$. All congruences below are modulo $n$.

We know that $\sigma(i) + \tau(i) \equiv b_i$ for all $i \neq i_1, i_2$. We construct the sequence $i_1, i_2, i_3, \dots$ as follows: for each $k \geq 2$, $i_{k+1}$ is the unique index such that

$$\sigma(i_{k-1}) + \tau(i_{k+1}) \equiv b_{i_k}. \tag{1}$$

Let $i_p = i_q$ be the repetition in the sequence with the smallest $q$. We claim that $p = 1$ or $p = 2$. Assume to the contrary that $p > 2$. Summing (1) for $k = p, p+1,$

$\dots, q-1$ and taking the equalities $\sigma(i_k)+\tau(i_k)=b_{i_k}$ for $i_k \neq i_1, i_2$ into account, we obtain $\sigma(i_{p-1})+\sigma(i_p)+\tau(i_{q-1})+\tau(i_q) \equiv b_p+b_{q-1}$. Since $i_q=i_p$, it follows that $\sigma(i_{p-1})+\tau(i_{q-1}) \equiv b_{q-1}$ and therefore $i_{p-1}=i_{q-1}$, a contradiction. Thus $p=1$ or $p=2$ as claimed.
Now we define the following permutations:

$$\begin{aligned}
\sigma'(i_k) &= \sigma(i_{k-1}) \text{ for } k=2,3,\dots,q-1 \text{ and } \sigma'(i_1)=\sigma(i_{q-1}),\\
\tau'(i_k) &= \tau(i_{k+1}) \text{ for } k=2,3,\dots,q-1 \text{ and } \tau'(i_1)=\begin{cases}\tau(i_2) \text{ if } p=1,\\ \tau(i_1) \text{ if } p=2;\end{cases}\\
\sigma'(i) &= \sigma(i) \text{ and } \tau'(i)=\tau(i) \quad \text{for } i \notin \{i_1,\dots,i_{q-1}\}.
\end{aligned}$$

Permutations $\sigma'$ and $\tau'$ have the desired property. Indeed, $\sigma'(i)+\tau'(i)=b_i$ obviously holds for all $i \neq i_1$, but then it must also hold for $i=i_1$.

13. For every green diagonal $d$, let $C_d$ denote the number of green–red intersection points on $d$. The task is to find the maximum possible value of the sum $\sum_d C_d$ over all green diagonals.
Let $d_i$ and $d_j$ be two green diagonals and let the part of polygon $M$ lying between $d_i$ and $d_j$ have $m$ vertices. There are at most $n-m-1$ red diagonals intersecting both $d_i$ and $d_j$, while each of the remaining $m-2$ diagonals meets at most one of $d_i, d_j$. It follows that

$$C_{d_i}+C_{d_j} \leq 2(n-m-1)+(m-2)=2n-m-4. \tag{1}$$

We now arrange the green diagonals in a sequence $d_1, d_2, \dots, d_{n-3}$ as follows. It is easily seen that there are two green diagonals $d_1$ and $d_2$ that divide $M$ into two triangles and an $(n-2)$-gon; then there are two green diagonals $d_3$ and $d_4$ that divide the $(n-2)$-gon into two triangles and an $(n-4)$-gon, and so on. We continue this procedure until we end up with a triangle or a quadrilateral. Now, the part of $M$ between $d_{2k-1}$ and $d_{2k}$ has at least $n-2k$ vertices for $1 \leq k \leq r$, where $n-3=2r+e$, $e \in \{0,1\}$; hence, by (1), $C_{d_{2k-1}}+C_{d_{2k}} \leq n+2k-4$. Moreover, $C_{d_{n-3}} \leq n-3$. Summing yields

$$\begin{aligned}
C_{d_1}+C_{d_2}+\cdots+C_{d_{n-3}} &\leq \sum_{k=1}^{r}(n+2k-4)+e(n-3)\\
&= 3r^2+e(3r+1)=\left\lceil \frac{3}{4}(n-3)^2 \right\rceil.
\end{aligned}$$

This value is attained in the following example. Let $A_1A_2\dots A_n$ be the $n$-gon $M$ and let $l=\left[\frac{n}{2}\right]+1$. The diagonals $A_1A_i$, $i=3,\dots,l$, and $A_lA_j$, $j=l+2,\dots,n$ are colored green, whereas the diagonals $A_2A_i$, $i=l+1,\dots,n$, and $A_{l+1}A_j$, $j=3,\dots,l-1$ are colored red.
Thus the answer is $\lceil \frac{3}{4}(n-3)^2 \rceil$.

14. Let $F$ be the point of tangency of the incircle with $AC$ and let $M$ and $N$ be the respective points of tangency of $AB$ and $BC$ with the corresponding excircles. If $I$ is the incenter and $I_a$ and $P$ respectively the center and the tangency point with ray $AC$ of the excircle corresponding to $A$, we have $\frac{AI}{IL}=\frac{AI}{IF}=\frac{AI_a}{I_aP}=\frac{AI_a}{I_aN}$, which

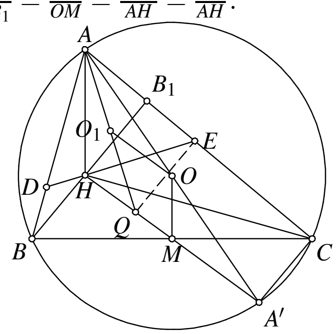
implies that $\triangle AIL \sim \triangle AI_aN$. Thus $L$ lies on $AN$, and analogously $K$ lies on $CM$. Define $x = AF$ and $y = CF$. Since $BD = BE$, $AD = BM = x$, and $CE = BN = y$, the condition $AB + BC = 3AC$ gives us $DM = y$ and $EN = x$. The triangles $CLN$ and $MKA$ are congruent since their altitudes $KD$ and $LE$ satisfy $DK = EL$, $DM = CE$, and $AD = EN$. Thus $\angle AKM = \angle CLN$, implying that $ACKL$ is cyclic.

15. Let $P$ be the fourth vertex of the rhombus $C_2A_1A_2P$. Since $\triangle C_2PC_1$ is equilateral, we easily conclude that $B_1B_2C_1P$ is also a rhombus. Thus $\triangle PB_1A_2$ is equilateral and $\angle(C_2A_1, C_1B_2) = \angle A_2PB_1 = 60^\circ$. It easily follows that $\triangle AC_1B_2 \cong \triangle BA_1C_2$ and consequently $AC_1 = BA_1$; similarly, $BA_1 = CB_1$. Therefore triangle $A_1B_1C_1$ is equilateral. Now it follows from $B_1B_2 = B_2C_1$ that $A_1B_2$ bisects $\angle C_1A_1B_1$. Similarly, $B_1C_2$ and $C_1A_2$ bisect $\angle A_1B_1C_1$ and $\angle B_1C_1A_1$; hence $A_1B_2$, $B_1C_2$, $C_1A_2$ meet at the incenter of $A_1B_1C_1$, i.e. at the center of $ABC$.

16. Since $\angle ADL = \angle KBA = 180^\circ - \frac{1}{2}\angle BCD$ and $\angle ALD = \frac{1}{2}\angle AYD = \angle KAB$, triangles $ABK$ and $LDA$ are similar. Thus $\frac{BK}{BC} = \frac{BK}{AD} = \frac{AB}{DL} = \frac{DC}{DL}$, which together with $\angle LDC = \angle CBK$ gives us $\triangle LDC \sim \triangle CBK$. Therefore $\angle KCL = 360^\circ - \angle BCD - (\angle LCD + \angle KCB) = 360^\circ - \angle BCD - (\angle CKB + \angle KCB) = 180^\circ - \angle CBK$, which is constant.

17. To start with, we note that points $B, E, C$ are the images of $D, F, A$ respectively under the rotation around point $O$ for the angle $\omega = \angle DOB$, where $O$ is the intersection of the perpendicular bisectors of $AC$ and $BD$. Then $OE = OF$ and $\angle OFE = \angle OAC = 90 - \frac{\omega}{2}$; hence the points $A, F, R, O$ are on a circle and $\angle ORP = 180^\circ - \angle OFA$. Analogously, the points $B, E, Q, O$ are on a circle and $\angle OQP = 180^\circ - \angle OEB = \angle OEC = \angle OFA$. This shows that $\angle ORP = 180^\circ - \angle OQP$, i.e. the point $O$ lies on the circumcircle of $\triangle PQR$, thus being the desired point.

18. Let $O$ and $O_1$ be the circumcenters of triangles $ABC$ and $ADE$, respectively. It is enough to show that $HM \parallel OO_1$. Let $AA'$ be the diameter of the circumcircle of $ABC$. We note that if $B_1$ is the foot of the altitude from $B$, then $HE$ bisects $\angle CHB_1$. Since the triangles $COM$ and $CHB_1$ are similar (indeed, $\angle CHB = \angle COM = \angle A$), we have $\frac{CE}{EB_1} = \frac{CH}{HB_1} = \frac{CO}{OM} = \frac{2CO}{AH} = \frac{A'A}{AH}$.




Thus, if $Q$ is the intersection point of the bisector of $\angle A'AH$ with $HA'$, we obtain $\frac{CE}{EB_1} = \frac{A'Q}{QH}$, which together with $A'C \perp AC$ and $HB_1 \perp AC$ gives us $QE \perp AC$. Analogously, $QD \perp AB$. Therefore $AQ$ is a diameter of the circumcircle of $\triangle ADE$ and $O_1$ is the midpoint of $AQ$. It follows that $OO_1$ is the line passing through the midpoints of $AQ$ and $AA'$; hence $OO_1 \| HM$.

*Second solution*. We again prove that $OO_1 \parallel HM$. Since $AA' = 2AO$, it suffices to prove $AQ = 2AO_1$.

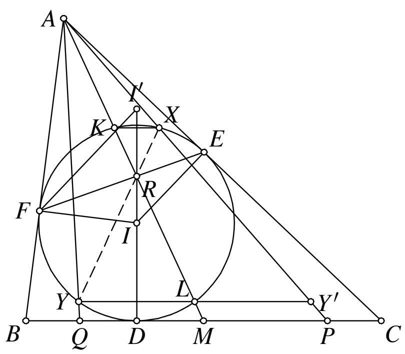
Elementary calculations of angles give us $\angle ADE = \angle AED = 90^\circ - \frac{\alpha}{2}$. Applying the law of sines to $\triangle DAH$ and $\triangle EAH$ we now have $DE = DH + EH = \frac{AH\cos\beta}{\cos\frac{\alpha}{2}} + \frac{AH\cos\gamma}{\cos\frac{\alpha}{2}}$. Since $AH = 2OM = 2R\cos\alpha$, we obtain

$$AO_1 = \frac{DE}{2\sin\alpha} = \frac{AH(\cos\beta+\cos\gamma)}{2\sin\alpha\cos\frac{\alpha}{2}} = \frac{2R\cos\alpha\sin\frac{\alpha}{2}\cos(\frac{\beta-\gamma}{2})}{\sin\alpha\cos\frac{\alpha}{2}}.$$

We now calculate $AQ$. Let $N$ be the intersection of $AQ$ with the circumcircle. Since $\angle NAO = \frac{\beta-\gamma}{2}$, we have $AN = 2R\cos(\frac{\beta-\gamma}{2})$. Noting that $\triangle QAH \sim \triangle QNM$ (and that $MN = R - OM$), we have

$$AQ = \frac{AN\cdot AH}{MN+AH} = \frac{2R\cos(\frac{\beta-\gamma}{2})\cdot 2\cos\alpha}{1+\cos\alpha} = \frac{2R\cos(\frac{\beta-\gamma}{2})\cos\alpha}{\cos^2\frac{\alpha}{2}} = 2AO_1.$$

19. We denote by $D,E,F$ the points of tangency of the incircle with $BC,CA,AB$, respectively, by $I$ the incenter, and by $Y'$ the intersection of $AX$ and $LY$. Since $EF$ is the polar line to the point $A$ with respect to the incircle, it meets $AL$ at point $R$ such that $A,R;K,L$ are conjugate, i.e., $\frac{KR}{RL} = \frac{KA}{AL}$. Then $\frac{KX}{LY'} = \frac{KA}{AL} = \frac{KR}{RL} = \frac{KX}{L\overline{Y}}$ and therefore $LY = L\overline{Y}$, where $\overline{Y}$ is the intersection of $XR$ and $LY$. Thus showing that $LY = LY'$ (which is the same as showing that $PM = MQ$, i.e., $CP = QB$) is equivalent to showing that $XY$ contains $R$. Since $XKYL$ is an inscribed trapezoid, it is enough to show that $R$ lies on its axis of symmetry, that is, $DI$.




Since $AM$ is the median, the triangles $ARB$ and $ARC$ have equal areas, and since $\angle(RF,AB) = \angle(RE,AC)$ we have that $1 = \frac{S_{\triangle ABR}}{S_{\triangle ACR}} = \frac{(AB\cdot FR)}{(AC\cdot ER)}$. Hence $\frac{AB}{AC} = \frac{ER}{FR}$. Let $I'$ be the point of intersection of the line through $F$ parallel to $IE$ with the line $IR$. Then $\frac{FI'}{EI} = \frac{FR}{RE} = \frac{AC}{AB}$ and $\angle I'FI = \angle BAC$ (angles with orthogonal rays). Thus the triangles $ABC$ and $FII'$ are similar, implying that $\angle FII' = \angle ABC$. Since $\angle FID = 180^\circ - \angle ABC$, it follows that $R, I$, and $D$ are collinear.

20. We shall prove the inequalities $p(ABC) \geq 2p(DEF)$ and $p(PQR) \geq \frac{1}{2}p(DEF)$. The statement of the problem will immediately follow.
Let $D_b$ and $D_c$ be the reflections of $D$ in $AB$ and $AC$, and let $A_1,B_1,C_1$ be the midpoints of $BC,CA,AB$, respectively. It is easy to see that $D_b,F,E,D_c$ are collinear. Hence $p(DEF) = D_bF + FE + ED_c = D_bD_c \leq D_bC_1 + C_1B_1 + B_1D_c = \frac{1}{2}(AB+BC+CA) = \frac{1}{2}p(ABC)$.
To prove the second inequality we observe that $P$, $Q$, and $R$ are the points of tangency of the excircles with the sides of $\triangle DEF$. Let $FQ = ER = x$, $DR = FP = y$, and $DQ = EP = z$, and let $\delta, \varepsilon, \varphi$ be the angles of $\triangle DEF$ at $D,E,F$,

respectively. Let $Q'$ and $R'$ be the projections of $Q$ and $R$ onto $EF$, respectively. Then $QR \geq Q'R' = EF - FQ' - R'E = EF - x(\cos\varphi + \cos\varepsilon)$. Summing this with the analogous inequalities for $FD$ and $DE$, we obtain

$$p(PQR) \geq p(DEF) - x(\cos\varphi + \cos\varepsilon) - y(\cos\delta + \cos\varphi) - z(\cos\delta + \cos\varepsilon).$$

Assuming without loss of generality that $x \leq y \leq z$, we also have $DE \leq FD \leq FE$ and consequently $\cos\varphi + \cos\varepsilon \geq \cos\delta + \cos\varphi \geq \cos\delta + \cos\varepsilon$. Now Chebyshev's inequality gives us $p(PQR) \geq p(DEF) - \frac{2}{3}(x+y+z)(\cos\varepsilon + \cos\varphi + \cos\delta) \geq p(DEF) - (x+y+z) = \frac{1}{2}p(DEF)$, where we used $x+y+z = \frac{1}{2}p(DEF)$ and the fact that the sum of the cosines of the angles in a triangle does not exceed $\frac{3}{2}$. This finishes the proof.

21. We will show that 1 is the only such number. It is sufficient to prove that for every prime number $p$ there exists some $a_m$ such that $p \mid a_m$. For $p = 2, 3$ we have $p \mid a_2 = 48$. Assume now that $p > 3$. Applying Fermat's theorem, we have

$$6a_{p-2} = 3 \cdot 2^{p-1} + 2 \cdot 3^{p-1} + 6^{p-1} - 6 \equiv 3 + 2 + 1 - 6 = 0 \pmod{p}.$$

Hence $p \mid a_{p-2}$, i.e. $\gcd(p, a_{p-2}) = p > 1$. This completes the proof.

22. It immediately follows from the condition of the problem that all the terms of the sequence are distinct. We also note that $|a_i - a_n| \leq n - 1$ for all integers $i, n$ where $i < n$, because if $d = |a_i - a_n| \geq n$ then $\{a_1, \dots, a_d\}$ contains two elements congruent to each other modulo $d$, which is a contradiction. It easily follows by induction that for every $n \in \mathbb{N}$ the set $\{a_1, \dots, a_n\}$ consists of consecutive integers. Thus, if we assumed that some integer $k$ did not appear in the sequence $a_1, a_2, \dots$, the same would have to hold for all integers either larger or smaller than $k$, which contradicts the condition that infinitely many positive and negative integers appear in the sequence. Thus, the sequence contains all integers.

23. Let us consider the polynomial

$$P(x) = (x+a)(x+b)(x+c) - (x-d)(x-e)(x-f) = Sx^2 + Qx + R,$$

where $Q = ab + bc + ca - de - ef - fd$ and $R = abc + def$.
Since $S \mid Q, R$, it follows that $S \mid P(x)$ for every $x \in \mathbb{Z}$. Hence, $S \mid P(d) = (d+a)(d+b)(d+c)$. Since $S > d+a, d+b, d+c$ and thus cannot divide any of them, it follows that $S$ must be composite.

24. We will show that $n$ has the desired property if and only if it is prime.
For $n = 2$ we can take only $a = 1$. For $n > 2$ and even, $4 \mid n!$, but $a^n + 1 \equiv 1, 2 \pmod 4$, which is impossible. Now we assume that $n$ is odd. Obviously $(n!-1)^n + 1 \equiv (-1)^n + 1 = 0 \pmod{n!}$. If $n$ is composite and $d$ its prime divisor, then $\left(\frac{n!}{d} - 1\right)^n + 1 = \sum_{k=1}^{n} \binom{n}{k} \frac{n!^k}{d^k}$, where each summand is divisible by $n!$ because $d^2 \mid n!$; therefore $n!$ divides $\left(\frac{n!}{d} - 1\right)^n + 1$. Thus, all composite numbers are ruled out.
It remains to show that if $n$ is an odd prime and $n! \mid a^n + 1$, then $n! \mid a + 1$, and therefore $a = n! - 1$ is the only relevant value for which $n! \mid a^n + 1$. Consider any

prime number $p \le n$. If $p \mid \frac{a^n+1}{a+1}$, we have $p \mid (-a)^n - 1$ and by Fermat's theorem $p \mid (-a)^{p-1} - 1$. Therefore $p \mid (-a)^{(n,p-1)} - 1 = -a - 1$, i.e. $a \equiv -1 \pmod{p}$. But then $\frac{a^n+1}{a+1} = a^{n-1} - a^{n-2} + \cdots - a + 1 \equiv n \pmod{p}$, implying that $p = n$. It follows that $\frac{a^n+1}{a+1}$ is coprime to $(n-1)!$ and consequently $(n-1)!$ divides $a+1$. Moreover, the above consideration shows that $n$ must divide $a+1$. Thus $n! \mid a+1$ as claimed. This finishes our proof.

25. We will use the abbreviation HD to denote a "highly divisible integer." Let $n = 2^{\alpha_2(n)} 3^{\alpha_3(n)} \cdots p^{\alpha_p(n)}$ be the factorization of $n$ into primes. We have $d(n) = (\alpha_2(n)+1)\cdots(\alpha_p(n)+1)$. We start with the following two lemmas.

*Lemma 1*. If $n$ is an HD and $p, q$ primes with $p^k < q^l$ $(k, l \in \mathbb{N})$, then

$$k\alpha_q(n) \le l\alpha_p(n) + (k+1)(l-1).$$

*Proof*. The inequality is trivial if $\alpha_q(n) < l$. Suppose that $\alpha_q(n) \ge l$. Then $np^k/q^l$ is an integer less than $q$, and $d(np^k/q^l) < d(n)$, which is equivalent to $(\alpha_q(n)+1)(\alpha_p(n)+1) > (\alpha_q(n)-l+1)(\alpha_p(n)+k+1)$ implying the desired inequality.

*Lemma 2*. For each $p$ and $k$ there exist only finitely many HD's $n$ such that $\alpha_p(n) \le k$.

*Proof*. It follows from Lemma 1 that if $n$ is an HD with $\alpha_p(n) \le k$, then $\alpha_q(n)$ is bounded for each prime $q$ and $\alpha_q(n) = 0$ for $q > p^{k+1}$. Therefore there are only finitely many possibilities for $n$.

We are now ready to prove both parts of the problem.

(a) Suppose that there are infinitely many pairs $(a, b)$ of consecutive HD's with $a \mid b$. Since $d(2a) > d(a)$, we must have $b = 2a$. In particular, $d(s) \le d(a)$ for all $s < 2a$. All but finitely many HD's $a$ are divisible by 2 and by $3^7$. Then $d(8a/9) < d(a)$ and $d(3a/2) < d(a)$ yield

$$(\alpha_2(a)+4)(\alpha_3(a)-1) < (\alpha_2(a)+1)(\alpha_3(a)+1) \Rightarrow 3\alpha_3(a) - 5 < 2\alpha_2(a),$$
$$\alpha_2(a)(\alpha_3(a)+2) \le (\alpha_2(a)+1)(\alpha_3(a)+1) \Rightarrow \alpha_2(a) \le \alpha_3(a)+1.$$

We now have $3\alpha_3(a) - 5 < 2\alpha_2(a) \le 2\alpha_3(a) + 2 \Rightarrow \alpha_3(a) < 7$, which is a contradiction.

(b) Assume for a given prime $p$ and positive integer $k$ that $n$ is the smallest HD with $\alpha_p \ge k$. We show that $\frac{n}{p}$ is also an HD. Assume the opposite, i.e., that there exists an HD $m < \frac{n}{p}$ such that $d(m) \ge d(\frac{n}{p})$. By assumption, $m$ must also satisfy $\alpha_p(m) + 1 \le \alpha_p(n)$. Then

$$d(mp) = d(m)\frac{\alpha_p(m)+2}{\alpha_p(m)+1} \ge d\left(\frac{n}{p}\right)\frac{\alpha_p(n)+1}{\alpha_p(n)} = d(n),$$

contradicting the initial assumption that $n$ is an HD (since $mp < n$). This proves that $\frac{n}{p}$ is an HD. Since this is true for every positive integer $k$, the proof is complete.

26. Assuming $b \neq a$, it trivially follows that $b > a$. Let $p > b$ be a prime number and let $n = (a+1)(p-1)+1$. We note that $n \equiv 1 \pmod{p-1}$ and $n \equiv -a \pmod{p}$. It follows that $r^n = r \cdot (r^{p-1})^{a+1} \equiv r \pmod{p}$ for every integer $r$. We now have $a^n + n \equiv a - a = 0 \pmod{p}$. Thus, $a^n + n$ is divisible by $p$, and hence by the condition of the problem $b^n + n$ is also divisible by $p$. However, we also have $b^n + n \equiv b - a \pmod{p}$, i.e., $p \mid b - a$, which contradicts $p > b$. Hence, it must follow that $b = a$. We note that $b = a$ trivially fulfills the conditions of the problem for all $a \in \mathbb{N}$.

27. Let $p$ be a prime and $k < p$ an even number. We note that $(p-k)!(k-1)! \equiv (-1)^{k-1}(p-k)!(p-k+1)\cdots(p-1) = (-1)^{k-1}(p-1)! \equiv 1 \pmod{p}$ by Wilson's theorem. Therefore

$$\begin{aligned}(k-1)!^n P((p-k)!) &= \textstyle\sum_{i=0}^{n} a_i [(k-1)!]^{n-i} [(p-k)!(k-1)!]^i \\ &\equiv \textstyle\sum_{i=0}^{n} a_i [(k-1)!]^{n-i} = S((k-1)!) \pmod{p},\end{aligned}$$

where $S(x) = a_n + a_{n-1}x + \cdots + a_0 x^n$. Hence $p \mid P((p-k)!)$ if and only if $p \mid S((k-1)!)$. Note that $S((k-1)!)$ depends only on $k$. Let $k > 2a_n + 1$. Then, $s = (k-1)!/a_n$ is an integer that is divisible by all primes smaller than $k$. Hence $S((k-1)!) = a_n b_k$ for some $b_k \equiv 1 \pmod{s}$. It follows that $b_k$ is divisible only by primes larger than $k$. For large enough $k$ we have $|b_k| > 1$. Thus for every prime divisor $p$ of $b_k$ we have $p \mid P((p-k)!)$.
It remains to select a large enough $k$ for which $|P((p-k)!)| > p$. We take $k = (q-1)!$, where $q$ is a large prime. All the numbers $k+i$ for $i = 1, 2, \dots, q-1$ are composite (by Wilson's theorem, $q \mid k+1$). Thus $p = k+q+r$, for some $r \geq 0$. We now have $|P((p-k)!)| = |P((q+r)!)| > (q+r)! > (q-1)! + q + r = p$, for large enough $q$, since $n = \deg P \geq 2$. This completes the proof.
*Remark*. The above solution actually also works for all linear polynomials $P$ other than $P(x) = x + a_0$. Nevertheless, these particular cases are easily handled. If $|a_0| > 1$, then $P(m!)$ is composite for $m > |a_0|$, whereas $P(x) = x+1$ and $P(x) = x-1$ are both composite for, say, $x = 5!$. Thus the condition $n \geq 2$ was redundant.

## 4.47 Solutions to the Shortlisted Problems of IMO 2006

1. If $a_0 \geq 0$ then $a_i \geq 0$ for each $i$ and $[a_{i+1}] \leq a_{i+1} = [a_i]\{a_i\} < [a_i]$ unless $[a_i] = 0$. Eventually 0 appears in the sequence $[a_i]$ and all subsequent $a_k$'s are 0.
Now suppose that $a_0 < 0$; then all $a_i \leq 0$. Suppose that the sequence never reaches 0. Then $[a_i] \leq -1$ and so $1 + [a_{i+1}] > a_{i+1} = [a_i]\{a_i\} > [a_i]$, so the sequence $[a_i]$ is nondecreasing and hence must be constant from some term on: $[a_i] = c < 0$ for $i \geq n$. The defining formula becomes $a_{i+1} = c\{a_i\} = c(a_i - c)$, which is equivalent to $b_{i+1} = cb_i$, where $b_i = a_i - \frac{c^2}{c-1}$. Since $(b_i)$ is bounded, we must have either $c = -1$, in which case $a_{i+1} = -a_i - 1$ and hence $a_{i+2} = a_i$, or $b_i = 0$ and thus $a_i = \frac{c^2}{c-1}$ for all $i \geq n$.

2. We use induction on $n$. We have $a_1 = 1/2$; assume that $n \geq 1$ and $a_1, \dots, a_n > 0$. The formula gives us $(n+1)\sum_{k=1}^{m} \frac{a_k}{m-k+1} = 1$. Writing this equation for $n$ and $n+1$ and subtracting yields
$$(n+2)a_{n+1} = \sum_{k=1}^{n} \left( \frac{n+1}{n-k+1} - \frac{n+2}{n-k+2} \right) a_k,$$
which is positive, as is the coefficient at each $a_k$.
*Remark*. Using techniques from complex analysis such as contour integrals, one can obtain the following formula for $n \geq 1$:
$$a_n = \int_1^{\infty} \frac{dx}{x^n(\pi^2 + \ln^2(x-1))} > 0.$$

3. We know that $c_n = \frac{\phi^{n-1} - \psi^{n-1}}{\phi - \psi}$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$ are the roots of $t^2 - t - 1$. Since $c_{n-1}/c_n \to -\psi$, taking $\alpha = \psi$ and $\beta = 1$ is a natural choice. For every finite set $J \subseteq \mathbb{N}$ we have
$$-1 = \sum_{n=0}^{\infty} \psi^{2n+1} < \psi x + y = \sum_{j \in J} \psi^{j-1} < \sum_{n=0}^{\infty} \psi^{2n} = \phi.$$
Thus $m = -1$ and $M = \phi$ is an appropriate choice. We now prove that this choice has the desired properties by showing that for any $x, y \in \mathbb{N}$ with $-1 < K = x\psi + y < \phi$, there is a finite set $J \subset \mathbb{N}$ such that $K = \sum_{j \in J} \psi^j$.
Given such $K$, there are sequences $i_1 \leq \cdots \leq i_k$ with $\psi^{i_1} + \cdots + \psi^{i_k} = K$ (one such sequence consists of $y$ zeros and $x$ ones). Consider all such sequences of minimum length $n$. Since $\psi^m + \psi^{m+1} = \psi^{m+2}$, these sequences contain no two consecutive integers. Order such sequences as follows: If $i_k = j_k$ for $1 \leq k \leq t$ and $i_t < j_t$, then $(i_r) \prec (j_r)$. Consider the smallest sequence $(i_r)_{r=1}^n$ in this ordering. We claim that its terms are distinct. Since $2\psi^2 = 1 + \psi^3$, replacing two equal terms $m, m$ by $m-2, m+1$ for $m \geq 2$ would yield a smaller sequence, so only 0 or 1 can repeat among the $i_r$. But $i_t = i_{t+1} = 0$ implies $\sum_r \psi^{i_r} > 2 + \sum_{k=0}^{\infty} \psi^{2k+3} = \phi$, while $i_t = i_{t+1} = 1$ similarly implies $\sum_r \psi^{i_r} < -1$, so both cases are impossible, proving our claim. Thus $J = \{i_1, \dots, i_n\}$ is a required set.

4. Since $\frac{ab}{a+b} = \frac{1}{4}\left(a+b-\frac{(a-b)^2}{a+b}\right)$, the left hand side of the desired inequality equals
$$A = \sum_{i<j} \frac{a_i a_j}{a_i + a_j} = \frac{n-1}{4}\sum_k a_k - \frac{1}{4}\sum_{i<j}\frac{(a_i - a_j)^2}{a_i + a_j}.$$
The righthand side of the inequality is equal to
$$B = \frac{n}{2}\frac{\sum a_i a_j}{\sum a_k} = \frac{n-1}{4}\sum_k a_k - \frac{1}{4}\sum_{i<j}\frac{(a_i - a_j)^2}{\sum a_k}.$$
Now $A \le B$ follows from the trivial inequality $\sum \frac{(a_i-a_j)^2}{a_i+a_j} \ge \sum \frac{(a_i-a_j)^2}{\sum a_k}$.

5. Let $x = \sqrt{b}+\sqrt{c}-\sqrt{a}$, $y = \sqrt{c}+\sqrt{a}-\sqrt{b}$, and $z = \sqrt{a}+\sqrt{b}-\sqrt{c}$. All of these numbers are positive because $a,b,c$ are sides of a triangle. Then $b+c-a = x^2 - \frac{1}{2}(x-y)(x-z)$ and
$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} = \sqrt{1-\frac{(x-y)(y-z)}{2x^2}} \le 1 - \frac{(x-y)(x-z)}{4x^2}.$$
Now it is enough to prove that
$$x^{-2}(x-y)(x-z) + y^{-2}(y-z)(y-x) + z^{-2}(z-x)(z-y) \ge 0,$$
which directly follows from Schur's inequality.

6. Assume, without loss of generality, that $a \ge b \ge c$. The lefthand side of the inequality equals $L = (a-b)(b-c)(a-c)(a+b+c)$. From $(a-b)(b-c) \le \frac{1}{4}(a-c)^2$ we get $L \le \frac{1}{4}(a-c)^3|a+b+c|$. The inequality $(a-c)^2 \le 2(a-b)^2 + 2(b-c)$ implies $(a-c)^2 \le \frac{2}{3}[(a-b)^2+(b-c)^2+(a-c)^2]$. Therefore
$$L \le \frac{\sqrt{2}}{2}\left(\frac{(a-b)^2+(b-c)^2+(a-c)^2}{3}\right)^{3/2}(a+b+c).$$
Finally, the mean inequality gives us
$$\begin{aligned} L &\le \frac{\sqrt{2}}{2}\left(\frac{(a-b)^2+(b-c)^2+(a-c)^2+(a+b+c)^2}{4}\right)^2 \\ &= \frac{9\sqrt{2}}{32}(a^2+b^2+c^2)^2. \end{aligned}$$
Equality is attained if and only if $a-b = b-c$ and $(a-b)^2+(b-c)^2+(a-c)^2 = 3(a+b+c)^2$, which leads to $a = \left(1+\frac{3}{\sqrt{2}}\right)b$ and $c = \left(1-\frac{3}{\sqrt{2}}\right)b$. Thus $M = \frac{9\sqrt{2}}{32}$.
*Second solution*. We have $L = |(a-b)(b-c)(c-a)(a+b+c)|$. Without loss of generality, assume that $a+b+c = 1$ (the case $a+b+c=0$ is trivial). The monic cubic polynomial with roots $a-b$, $b-c$, and $c-a$ is of the form

$$P(x) = x^3 + qx + r, \quad q = \frac{1}{2} - \frac{3}{2}(a^2 + b^2 + c^2), \;\; r = -(a-b)(b-c)(c-a).$$

Then $M^2 = \max r^2 / \left(\frac{1-2q}{3}\right)^4$. Since $P(x)$ has three real roots, its discriminant $(q/3)^3 + (r/2)^2$ must be positive, so $r^2 \geq -\frac{4}{27}q^3$. Thus $M^2 \leq f(q) = -\frac{4}{27}q^3 / \left(\frac{1-2q}{3}\right)^4$. The function $f$ attains its maximum $3^4/2^9$ at $q = -3/2$, so $M \leq \frac{9\sqrt{2}}{32}$. The case of equality is easily computed.

*Third solution*. Assume that $a^2 + b^2 + c^2 = 1$ and write $u = (a+b+c)/\sqrt{3}$, $v = (a + \varepsilon b + \varepsilon^2 c)/\sqrt{3}$, $w = (a + \varepsilon^2 b + \varepsilon c)/\sqrt{3}$, where $\varepsilon = e^{2\pi i/3}$. Then analogous formulas hold for $a, b, c$ in terms of $u, v, w$, from which one directly obtains $|u|^2 + |v|^2 + |w|^2 = a^2 + b^2 + c^2 = 1$ and

$$a + b + c = \sqrt{3}u, \quad |a-b| = |v - \varepsilon w|, \;\; |a-c| = |v - \varepsilon^2 w|, \;\; |b-c| = |v-w|.$$

Thus $L = \sqrt{3}|u||v^3 - w^3| \leq \sqrt{3}|u|(|v|^3 + |w|^3) \leq \sqrt{\frac{3}{2}|u|^2(1-|u|^2)^3} \leq \frac{9\sqrt{2}}{32}$. It is easy to trace back $a, b, c$ to the equality case.

7. (a) We show that for $n = 2^k$ all lamps will be switched on in $n-1$ steps and off in $n$ steps. For $k = 1$ the statement is true. Suppose it holds for some $k$ and let $n = 2^{k+1}$; define $L = \{L_1, \dots, L_{2^k}\}$ and $R = \{L_{2^k+1}, \dots, L_{2^{k+1}}\}$. The first $2^k - 1$ steps are performed without any influence on or from the lamps from $R$; thus after $2^k - 1$ steps the lamps in $L$ are on and those from $R$ are off. After the $2^k$th step, $L_{2^k}$ and $L_{2^k+1}$ are on and the other lamps are off. Notice that from now on, $L$ and $R$ will be symmetric (i.e., $L_i$ and $L_{2^{k+1}-i}$ will have the same state) and will never influence each other. Since $R$ starts with only the leftmost lamp on, in $2^k$ steps all its lamps will be off. The same will happen to $L$. There are $2^k + 2^k = 2^{k+1}$ steps in total.

   (b) We claim that for $n = 2^k + 1$ the lamps cannot be switched off. After the first step, only $L_1$ and $L_2$ are on. According to (a), after $2^k - 1$ steps all lamps but $L_n$ will be on, so after the $2^k$th step all lamps will be off except for $L_{n-1}$ and $L_n$. Since this position is symmetric to the one after the first step, the procedure will never end.

8. We call a triangle *odd* if it has two odd sides. To any odd isosceles triangle $A_iA_jA_k$ we assign a pair of sides of the 2006-gon. We may assume that $k - j = j - i > 0$ is odd. A side of the 2006-gon is said to *belong* to triangle $A_iA_jA_k$ if it lies on the polygonal line $A_iA_{i+1}\dots A_k$. At least one of the odd number of sides $A_iA_{i+1}, \dots, A_{j-1}A_j$ and at least one of the sides $A_jA_{j+1}, \dots, A_{k-1}A_k$ do not belong to any other odd isosceles triangle; assign those two sides to $\triangle A_iA_jA_k$. This ensures that every two assigned pairs are disjoint; therefore there are at most 1003 odd isosceles triangles.

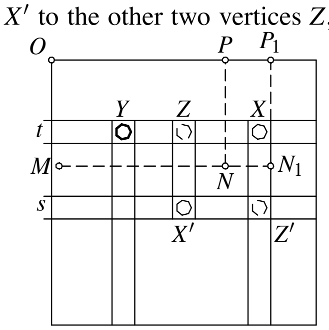
   An example with 1003 odd isosceles triangles can be attained when the diagonals $A_{2k}A_{2k+2}$ are drawn for $k = 0, \dots, 1002$, where $A_0 = A_{2006}$.

9. The number $c(P)$ of points inside $P$ is equal to $n - a(P) - b(P)$, where $n = |S|$. Writing $y = 1 - x$, the considered sum becomes

$$\sum_P x^{a(P)} y^{b(P)} (x+y)^{c(P)} = \sum_P \sum_{i=0}^{c(P)} \binom{c(P)}{i} x^{a(P)+i} y^{b(P)+c(P)-i}$$
$$= \sum_P \sum_{k=a(P)}^{a(P)+c(P)} \binom{c(P)}{k-a(P)} x^k y^{n-k}.$$

Here the coefficient at $x^k y^{n-k}$ is the sum $\sum_P \binom{c(P)}{k-a(P)}$, which equals the number of pairs $(P,Z)$ of a convex polygon $P$ and a $k$-element subset $Z$ of $S$ whose convex hull is $P$, and is thus equal to $\binom{n}{k}$. Now the required statement immediately follows.

10. Denote by $S_{\mathscr{A}}(R)$ the number of strawberries of arrangement $\mathscr{A}$ inside rectangle $R$. We write $\mathscr{A} \le \mathscr{B}$ if for every rectangle $Q$ containing the top left corner $O$ we have $S_{\mathscr{B}}(Q) \ge S_{\mathscr{A}}(Q)$. In this ordering, every switch transforms an arrangement to a larger one. Since the number of arrangements is finite, it is enough to prove that whenever $\mathscr{A} < \mathscr{B}$ there is a switch taking $\mathscr{A}$ to $\mathscr{C}$ with $\mathscr{C} \le \mathscr{B}$. Consider the highest row $t$ of the cake that differs in $\mathscr{A}$ and $\mathscr{B}$; let $X$ and $Y$ be the positions of the strawberries in $t$ in $\mathscr{A}$ and $\mathscr{B}$ respectively. Clearly $Y$ is to the left from $X$ and the strawberry of $\mathscr{A}$ in the column of $Y$ is below $Y$. Now consider the highest strawberry $X'$ of $\mathscr{A}$ below $t$ whose column is between $X$ and $Y$ (including $Y$). Let $s$ be the row of $X'$. Now switch $X, X'$ to the other two vertices $Z, Z'$ of the corresponding rectangle, obtaining an arrangement $\mathscr{C}$. We claim that $\mathscr{C} \le \mathscr{B}$. It is enough to verify that $S_{\mathscr{C}}(Q) \le S_{\mathscr{B}}(Q)$ for those rectangles $Q = OMNP$ with $N$ lying inside $XZX'Z'$. Let $Q' = OMN_1P_1$ be the smallest rectangle containing $X$. Our choice of $s$ ensures that $S_{\mathscr{C}}(Q) = S_{\mathscr{A}}(Q') \ge S_{\mathscr{B}}(Q') \ge S_{\mathscr{B}}(Q)$, as claimed.




11. Let $q$ be the largest integer such that $2^q \mid n$. We prove that an $(n,k)$-tournament exists if and only if $k < 2^q$.
The first $l$ rounds of an $(n,k)$-tournament form an $(n,l)$-tournament. Thus it is enough to show that an $(n, 2^q-1)$-tournament exists and an $(n, 2^q)$-tournament does not.
If $n = 2^q$, we can label the contestants and rounds by elements of the additive group $\mathbb{Z}_2^q$. If contestants $x$ and $x+j$ meet in the round labeled $j$, it is easy to verify the conditions. If $n = 2^q p$, we can divide the contestants into $p$ disjoint groups of $2^q$ and perform a $(2^q, 2^q-1)$-tournament in each, thus obtaining an $(n, 2^q-1)$-tournament.
For the other direction let $\mathscr{G}_i$ be the graph of players with edges between any two players who met in the first $i$ rounds. We claim that the size of each connected component of $\mathscr{G}_i$ is a power of 2. For $i = 1$ this is obvious; assume that it holds for $i$. Suppose that the components $C$ and $D$ merge in the $(i+1)$th round. Then

some $c \in C$ and $d \in D$ meet in this round. Moreover, each player in $C$ meets a player in $D$. Indeed, for every $c' \in C$ there is a path $c = c_0, c_1, \dots, c_k = c'$ with $c_j c_{j+1} \in \mathscr{G}_i$; then if $d_j$ is the opponent of $c_j$ in the $(i+1)$th round, condition (ii) shows that each $d_j d_{j+1}$ belongs to $\mathscr{G}_i$, so $d_k \in D$. Analogously, all players in $D$ meet players in $C$, so $|C| = |D|$, proving our claim. Now if there are $2^q$ rounds, every component has size at least $2^q + 1$ and is thus divisible by $2^{q+1}$, which is impossible if $2^{q+1} \nmid n$.

12. Let $U$ and $D$ be the sets of upward and downward unit triangles, respectively. Two triangles are *neighbors* if they form a diamond. For $A \subseteq D$, denote by $F(A)$ the set of neighbors of the elements of $A$.
If a holey triangle can be tiled with diamonds, in every upward triangle of side $l$ there are $l^2$ elements of $D$, so there must be at least as many elements of $U$ and at most $l$ holes.
Now we pass to the other direction. It is enough to show the condition (ii) of the marriage theorem: For every set $X \subset D$ we have $|F(X)| \geq |X|$. Assume the contrary, that $|F(X)| < |X|$ for some set $X$. Note that any two elements of $D$ with a common neighbor must share a vertex; this means that we can focus on connected sets $X$. Consider an upward triangle of side 3. It contains three elements of $D$; if two of them are in $X$, adding the third one to $X$ increases $F(X)$ by at most 1, so $|F(X)| < |X|$ still holds. Continuing this procedure, we will end up with a set $X$ forming an upward subtriangle of $T$ and satisfying $|F(X)| < |X|$, which contradicts the conditions of the problem. This contradiction proves that $|F(X)| \geq |X|$ for every set $X$, and an application of the Hall's marriage theorem establishes the result.

13. Consider a polyhedron $\mathscr{P}$ with $v$ vertices, $e$ edges, and $f$ faces. Consider the map $\sigma$ to the unit sphere $S$ taking each vertex, edge, or face $x$ of $\mathscr{P}$ to the set of outward unit normal vectors (i.e., points on $S$) to the support planes of $\mathscr{P}$ containing $x$. Thus $\sigma$ maps faces to points on $S$, edges to shorter arcs of big circles connecting some pairs of these points, and vertices to spherical regions formed by these arcs. These points, arcs, and regions on $S$ form a "spherical polyhedron" $\mathscr{G}$.
We now translate the conditions of the problem into the language of $\mathscr{G}$. Denote by $\overline{x}$ the image of $x$ through reflection with the center in the center of $S$. No edge of $\mathscr{P}$ being parallel to another edge or face means that the big circle of any edge $e$ of $\mathscr{G}$ does not contain any vertex $V$ nonincident to $e$. Also note that vertices $A$ and $B$ of $\mathscr{P}$ are antipodal if and only if $\sigma(A)$ and $\overline{\sigma(B)}$ intersect, and that the midpoints of edges $a$ and $b$ are antipodal if and only if $\sigma(a)$ and $\overline{\sigma(b)}$ intersect. Consider the union $\mathscr{F}$ of $\mathscr{G}$ and $\overline{\mathscr{G}}$. The faces of $\mathscr{F}$ are the intersections of faces of $\mathscr{G}$ and $\overline{\mathscr{G}}$, so their number equals $2A$. Similarly, the edges of $\mathscr{G}$ and $\overline{\mathscr{G}}$ have $2B$ intersections, so $\mathscr{F}$ has $2e+4B$ edges and $2f+2B$ vertices. Now Euler's theorem for $\mathscr{F}$ gives us $2e+4B+2 = 2A+2f+2B$, and therefore $A - B = e - f + 1$.

14. The condition of the problem implies that $\angle PBC + \angle PCB = 90^\circ - \alpha/2$, i.e., $\angle BPC = 90^\circ + \alpha/2 = \angle BIC$. Thus $P$ lies on the circumcircle $\omega$ of $\triangle BCI$. It is

well known that the center $M$ of $\omega$ is the second intersection of $AI$ with the circumcircle of $\triangle ABC$. Therefore $AP \geq AM - MP = AM - MI = AI$, with equality if and only if $P \equiv I$.

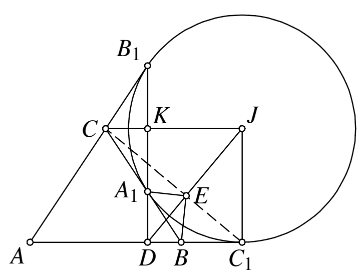
15. The relation $AK/KB = DL/LC$ implies that $AD$, $BC$, and $KL$ have a common point $O$. Moreover, since $\angle APB = 180^\circ - \angle ABC$ and $\angle DQC = 180^\circ - \angle BCD$, line $BC$ is tangent to the circles $APB$ and $CQD$. These two circles are homothetic with respect to $O$, so if $OP$ meets circle $APB$ again at $P'$, we have $\angle PQC = \angle PP'B = \angle PBC$, showing that $P, Q, B, C$ lie on a circle.

16. Let the diagonals $AC$ and $BD$ meet at $Q$ and $AD$ and $CE$ meet at $R$. The quadrilaterals $ABCD$ and $ACDE$ are similar, so $AQ/QC = AR/RD$. Now if $AP$ meets $CD$ at $M$, Ceva's theorem gives us $\frac{CM}{MD} = \frac{CQ}{QA} \cdot \frac{AR}{RD} = 1$.

17. Let $M$ be the point on $AC$ such that $JM \parallel KL$. It is enough to prove that $AM = 2AL$.
From $\angle BDA = \alpha$ we obtain that $\angle JDM = 90^\circ - \frac{\alpha}{2} = \angle KLA = \angle JMD$; hence $JM = JD$, and the tangency point of the incircle of $\triangle BCD$ with $CD$ is the midpoint $T$ of segment $MD$. Therefore, $DM = 2DT = BD + CD - BC = AB - BC + CD$, which gives us

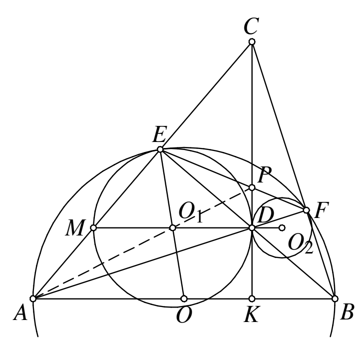
$$AM = AD + DM = AC + AB - BC = 2AL.$$

18. Assume that $A_1B_1$ and $CJ$ intersect at $K$. Then $JK$ is parallel and equal to $C_1D$ and $DC_1/C_1J = JK/JB_1 = JB_1/JC = C_1J/JC$, so the right triangles $DC_1J$ and $C_1JC$ are similar; hence $C_1C \perp DJ$. Thus $E$ belongs to $CC_1$. The points $A_1$, $B_1$, and $E$ lie on the circle with diameter $CJ$. Therefore $\angle DBA_1 = \angle A_1CJ = \angle A_1ED$, implying that $BEA_1D$ is cyclic; hence $\angle A_1EB = 90^\circ$. Likewise, $ADEB_1$ is cyclic because $\angle EB_1A = \angle EJC = \angle EDC_1$, so $\angle AEB_1 = 90^\circ$.




*Second solution.* The segments $JA_1$, $JB_1$, $JC_1$ are tangent to the circles with diameters $A_1B$, $AB_1$, $C_1D$. Since $JA_1^2 = JB_1^2 = JC_1^2 = JD \cdot JE$, $E$ lies on the first two circles (with diameters $A_1B$ and $AB_1$), so $\angle AEB_1 = \angle A_1EB = 90^\circ$.

19. The homothety with center $E$ mapping $\omega_1$ to $\omega$ maps $D$ to $B$, so $D$ lies on $BE$; analogously, $D$ lies on $AF$. Let $AE$ and $BF$ meet at point $C$. The lines $BE$ and $AF$ are altitudes of triangle $ABC$, so $D$ is the orthocenter and $C$ lies on $t$. Let the line through $D$ parallel to $AB$ meet $AC$ at $M$. The centers $O_1$ and $O_2$ are the midpoints of $DM$ and $DN$ respectively.



We have thus reduced the problem to a classical triangle geometry problem: If $CD$ and $EF$ intersect at $P$, we should prove that points $A$, $O_1$ and $P$ are collinear (analogously, so are $B$, $O_2$, $P$). By Menelaus's theorem for triangle $CDM$, this is equivalent to $\frac{CA}{AM} = \frac{CP}{PD}$, which is again equivalent to $\frac{CK}{KD} = \frac{CP}{PD}$ (because $DM \parallel AB$), where $K$ is the foot of the altitude from $C$ to $AB$. The last equality immediately follows from the fact that the pairs $C,D$; $P,K$ are harmonically adjoint.

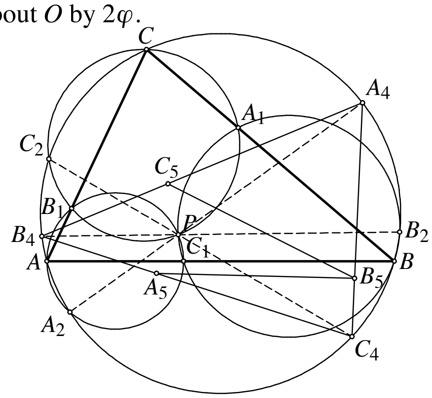
20. Let $I$ be the incenter of $\triangle ABC$. It is well known that $T_aT_c$ and $T_aT_b$ are the perpendicular bisectors of the segments $BI$ and $CI$ respectively. Let $T_aT_b$ meet $AC$ at $P$ and $\omega_b$ at $U$, and let $T_aT_c$ meet $AB$ at $Q$ and $\omega_c$ at $V$. We have $\angle BIQ = \angle IBQ = \angle IBC$, so $IQ \parallel BC$; similarly $IP \parallel BC$. Hence $PQ$ is the line through $I$ parallel to $BC$.
The homothety from $T_b$ mapping $\omega_b$ to the circumcircle $\omega$ of $ABC$ maps the tangent $t$ to $\omega_b$ at $U$ to the tangent to $\omega$ at $T_a$ that is parallel to $BC$. It follows that $t \parallel BC$. Let $t$ meet $AC$ at $X$. Since $XU = XM_b$ and $\angle PUM_b = 90^\circ$, $X$ is the midpoint of $PM_b$. Similarly, the tangent to $\omega_c$ at $V$ meets $QM_c$ at its midpoint $Y$. But since $XY \parallel PQ \parallel M_bM_c$, points $U,X,Y,V$ are collinear, so $t$ coincides with the common tangent $p_a$. Thus $p_a$ runs midway between $I$ and $M_bM_c$. Analogous conclusions hold for $p_b$ and $p_c$, so these three lines form a triangle homothetic to the triangle $M_aM_bM_c$ from center $I$ in ratio $\frac{1}{2}$, which is therefore similar to the triangle $ABC$ in ratio $\frac{1}{4}$.

21. The following proposition is easy to prove:
*Lemma*. For an arbitrary point $X$ inside a convex quadrilateral $ABCD$, the circumcircles of triangles $ADX$ and $BCX$ are tangent at $X$ if and only if $\angle ADX + \angle BCX = \angle AXB$.
Let $Q$ be the second intersection point of the circles $ABP$ and $CDP$ (we assume $Q \not\equiv P$; the opposite case is similarly handled). It follows from the conditions of the problem that $Q$ lies inside quadrilateral $ABCD$ (since $\angle BCP + \angle BAP < 180^\circ$, $C$ is outside the circumcircle of $APB$; the same holds for $D$). If $Q$ is inside $\triangle APD$ (the other case is similar), we have $\angle BQC = \angle BQP + \angle PQC = \angle BAP + \angle CDP \leq 90^\circ$. Similarly, $\angle AQD \leq 90^\circ$. Moreover, $\angle ADQ + \angle BCQ = \angle ADP + \angle BCP = \angle APB = \angle AQB$ implies that circles $ADQ$ and $BCQ$ are tangent at $Q$. Therefore the interiors of the semicircles with diameters $AD$ and $BC$ are disjoint, and if $M$, $N$ are the midpoints of $AD$ and $BC$ respectively, we have $2MN \geq AD + BC$. On the other hand, $2MN \leq AB + CD$ because $\overrightarrow{BA} + \overrightarrow{CD} = 2\overrightarrow{MN}$, and the statement of the problem immediately follows.

22. We work with oriented angles modulo $180^\circ$. For two lines $a,b$ we denote by $\angle(l,m)$ the angle of counterclockwise rotation transforming $a$ to $b$; also, by $\angle ABC$ we mean $\angle(BA,BC)$.
It is well known that the circles $AB_1C_1$, $BC_1A_1$, and $CA_1B_1$ have a common point, say $P$. Let $O$ be the circumcenter of $ABC$. Set $\angle PB_1C = \angle PC_1A = \angle PA_1B = \varphi$. Let $A_2P, B_2P, C_2P$ meet the circle $ABC$ again at $A_4, B_4, C_4$, respectively. Since

$\angle A_4A_2A = \angle PA_2A = \angle PC_1A = \varphi$ and thus $\angle A_4OA = 2\varphi$ etc., $\triangle ABC$ is the image of $\triangle A_4B_4C_4$ under rotation $\mathscr{R}$ about $O$ by $2\varphi$.

Therefore $\angle(AB_4, PC_1) = \angle B_4AB + \angle AC_1P = \varphi - \varphi = 0$, so $AB_4 \parallel PC_1$. Let $PC_1$ intersect $A_4B_4$ at $C_5$; define $A_5, B_5$ analogously. Then $\angle B_4C_5P = \angle A_4B_4A = \varphi$, so $AB_4C_5C_1$ is an isosceles trapezoid with $BC_3 = AC_1 = B_4C_5$. Similarly, $AC_3 = A_4C_5$, so $C_3$ is the image of $C_5$ under $\mathscr{R}$; similar statements hold for $A_5, B_5$. Thus $\triangle A_3B_3C_3 \cong \triangle A_5B_5C_5$. It remains to show that $\triangle A_5B_5C_5 \sim \triangle A_2B_2C_2$.




We have seen that $\angle A_4B_5P = \angle B_4C_5P$, which implies that $P$ lies on the circle $A_4B_5C_5$. Analogously, $P$ lies on the circle $C_4A_5B_5$. Therefore

$$\begin{aligned}\angle A_2B_2C_2 &= \angle A_2B_2B_4 + \angle B_4B_2C_2 = \angle A_2A_4B_4 + \angle B_4C_4C_2 \\ &= \angle PA_4C_5 + \angle A_5C_4P = \angle PB_5C_5 + \angle A_5B_5P = \angle A_5B_5C_5,\end{aligned}$$

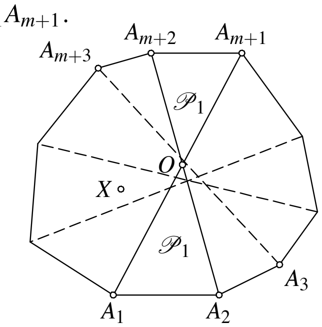
and similarly for the other angles, which is what we wanted.

23. Let $S_i$ be the area assigned to side $A_iA_{i+1}$ of polygon $\mathscr{P} = A_1 \dots A_n$ of area $S$. We start with the following auxiliary statement.

*Lemma.* At least one of the areas $S_1, \dots, S_n$ is not smaller than $2S/n$.

*Proof.* It suffices to prove the statement for even $n$. The case of odd $n$ will then follow immediately from this case applied to the degenerate $2n$-gon $A_1A_1' \dots A_nA_n'$, where $A_i'$ is the midpoint of $A_iA_{i+1}$.

Let $n = 2m$. For $i = 1, 2, \dots, m$, denote by $T_i$ the area of the region $\mathscr{P}_i$ inside the polygon bounded by the diagonals $A_iA_{m+i}$, $A_{i+1}A_{m+i+1}$ and the sides $A_iA_{i+1}$, $A_{m+i}A_{m+i+1}$. We observe that the regions $\mathscr{P}_i$ cover the entire polygon. Indeed, let $X$ be an arbitrary point inside the polygon, to the left (without loss of generality) of the ray $A_1A_{m+1}$. Then $X$ is to the right of the ray $A_{m+1}A_1$, so there is a $k$ such that $X$ is to the left of ray $A_kA_{k+m}$ and to the right of ray $A_{k+1}A_{k+m+1}$, i.e., $X \in \mathscr{P}_k$. It follows that $T_1 + \cdots + T_m \geq S$; hence at least one $T_i$ is not smaller than $2S/n$, say $T_1 \geq 2S/n$.




Let $O$ be the intersection point of $A_1A_{m+1}$ and $A_2A_{m+2}$, and let us assume without loss of generality that $S_{A_1A_2O} \geq S_{A_{m+1}A_{m+2}O}$ and $A_1O \geq OA_{m+1}$. Then required result now follows from

$$S_1 \geq S_{A_1A_2A_{m+2}} = S_{A_1A_2O} + S_{A_1A_{m+2}O} \geq S_{A_1A_2O} + S_{A_{m+1}A_{m+2}O} = T_1 \geq \frac{2S}{n}.$$

If, contrary to the assertion, $\sum \frac{S_i}{S} < 2$, we can choose rational numbers $q_i = 2m_i/N$ with $N = m_1 + \cdots + m_n$ such that $q_i > S_i/S$. However, considering the given polygon as a degenerate $N$-gon obtained by division of side $A_iA_{i+1}$ into $m_i$ equal parts for each $i$ and applying the lemma, we obtain $S_i/m_i \geq 2S/N$, i.e., $S_i/S \geq q_i$ for some $i$, a contradiction.
Equality holds if and only if $\mathscr{P}$ is centrally symmetric.
*Second solution*. We say that vertex $V$ is assigned to side $a$ of a convex (possibly degenerate) polygon $\mathscr{P}$ if the triangle determined by $a$ and $V$ has the maximum area $S_a$ among the triangles with side $a$ contained in $\mathscr{P}$. Define $\sigma(\mathscr{P}) = \sum_a S_a$ and $\delta(\mathscr{P}) = \sigma(\mathscr{P}) - 2[\mathscr{P}]$. We use induction on the number $n$ of pairwise nonparallel sides of $\mathscr{P}$ to show that $\delta(\mathscr{P}) \geq 0$ for every polygon $\mathscr{P}$. This is obviously true for $n = 2$, so let $n \geq 3$.
There exist two adjacent sides $AB$ and $BC$ whose respective assigned vertices $U$ and $V$ are distinct. Let the lines through $U$ and $V$ parallel to $AB$ and $BC$ respectively intersect at point $X$. Assume, without loss of generality, that there are no sides of $\mathscr{P}$ lying on $UX$ and $VX$. Call the sides and vertices of $\mathscr{P}$ lying within the triangle $UVX$ *passive* (excluding vertices $U$ and $V$). It is easy to see that no passive vertex is assigned to any side of $\mathscr{P}$ and that vertex $B$ is assigned to every passive side. Now replace all passive vertices of $\mathscr{P}$ by $X$, obtaining a polygon $\mathscr{P}'$. Vertex $B$ is assigned to sides $UX$ and $VX$ of $\mathscr{P}'$. Therefore the sum of areas assigned to passive sides increases by the area $S$ of the part of quadrilateral $BUXV$ lying outside $\mathscr{P}$; the other assigned areas do not change. Thus $\sigma$ increases by $S$. On the other hand, the area of the polygon also increases by $S$, so $\delta$ must decrease by $S$.
Note that the change from $\mathscr{P}$ to $\mathscr{P}'$ decreases the number of nonparallel sides. Thus by the inductive hypothesis we have $\delta(\mathscr{P}) \geq \delta(\mathscr{P}') \geq 0$.
*Third solution*. To each convex $n$-gon $\mathscr{P} = A_1A_2\dots A_n$ we assign a centrally symmetric $2n$-gon $\mathscr{Q}$, called the *associate* of $\mathscr{P}$, as follows. Attach the $2n$ vectors $\pm\overrightarrow{A_iA_{i+1}}$ at a common origin and label them $b_1, \dots, b_{2n}$ counterclockwise so that $b_{n+i} = -b_i$ for $1 \leq i \leq n$. Then take $\mathscr{Q}$ to be the polygon $B_1B_2\dots B_{2n}$ with $\overrightarrow{B_iB_{i+1}} = b_i$. Denote by $a_i$ the side of $\mathscr{P}$ corresponding to $b_i$ $(i = 1, \dots, n)$. The distance between the parallel sides $B_iB_{i+1}$ and $B_{n+i}B_{n+i+1}$ of $\mathscr{Q}$ equals twice the maximum height of $\mathscr{P}$ to the side $a_i$. Thus, if $O$ is the center of $\mathscr{Q}$, the area of $\triangle B_iB_{i+1}O$ $(i = 1, \dots, n)$ is exactly the area $S_i$ assigned to side $a_i$ of $\mathscr{P}$; therefore $[\mathscr{Q}] = 2\sum S_i$. It remains to show that $d(\mathscr{P}) = [\mathscr{Q}] - 4[\mathscr{P}] \geq 0$.

(i) Suppose that $\mathscr{P}$ has two parallel sides $a_i$ and $a_j$, where $a_j \geq a_i$, and remove from it the parallelogram $D$ determined by $a_i$ and a part of side $a_j$. We obtain a polygon $\mathscr{P}'$ with a smaller number of nonparallel sides. Then the associate of $\mathscr{P}'$ is obtained from $\mathscr{Q}$ by removing a parallelogram similar to $D$ in ratio 2 (and with area four times that of $D$); thus $d(\mathscr{P}') = d(\mathscr{P})$.

(ii) Suppose that there is a side $b_i$ $(i \leq n)$ of $\mathscr{Q}$ such that the sum of the angles at its endpoints is greater than $180°$. Extend the pairs of sides adjacent to $b_i$ and $b_{n+i}$ to their intersections $U$ and $V$, thus enlarging $\mathscr{Q}$ by two congruent triangles to a polygon $\mathscr{Q}'$. Then $\mathscr{Q}'$ is the associate of the polygon $\mathscr{P}'$

obtained from $\mathscr{P}$ by attaching a triangle congruent to $B_iB_{i+1}U$ to the side $a_i$. Therefore $d(\mathscr{P}')$ equals $d(\mathscr{P})$ minus twice the area of the attached triangle.

By repeatedly performing the operations (i) and (ii) to polygon $\mathscr{P}$ we will eventually reduce it to a parallelogram $E$, thereby decreasing the value of $d$. Since $d(E)=0$, it follows that $d(\mathscr{P})\ge 0$.

*Remark*. Polygon $\mathscr{Q}$ is the Minkowski sum of $\mathscr{P}$ and a polygon centrally symmetric to $\mathscr{P}$. Thus the inequality $[\mathscr{Q}]\ge 4[\mathscr{P}]$ is a direct consequence of the Brunn–Minkowski inequality.

24. Obviously $x\ge 0$. For $x=0$ the only solutions are $(0,\pm 2)$. Now let $(x,y)$ be a solution with $x>0$. Without loss of generality, assume that $y>0$. The equation rewritten as $2^x(1+2^{x+1})=(y-1)(y+1)$ shows that one of the factors $y\pm 1$ is divisible by 2 but not by 4 and the other by $2^{x-1}$ but not by $2^x$; hence $x\ge 3$. Thus $y=2^{x-1}m+\varepsilon$, where $m$ is odd and $\varepsilon=\pm 1$. Plugging this in the original equation and simplifying yields
$$2^{x-2}(m^2-8)=1-\varepsilon m. \tag{1}$$
Since $m=1$ is obviously impossible, we have $m\ge 3$ and hence $\varepsilon=-1$. Now (1) gives us $2(m^2-8)\le 1+m$, implying $m=3$, which leads to $x=4$ and $y=23$. Thus all solutions are $(0,\pm 2)$ and $(4,\pm 23)$.

25. If $x$ is rational, its digits repeat periodically starting at some point. If $n$ is the length of the period of $x$, the sequence $2,2^2,2^3,\dots$ is eventually periodic modulo $n$, so the corresponding digits of $x$ (i.e., the digits of $y$) also make an eventually periodic sequence, implying that $y$ is rational.

26. Consider $g(n)=[\frac{n}{1}]+[\frac{n}{2}]+\cdots+[\frac{n}{n}]=nf(n)$ and define $g(0)=0$. Since for any $k$ the difference $[\frac{n}{k}]-[\frac{n-1}{k}]$ equals 1 if $k$ divides $n$ and 0 otherwise, we obtain that $g(n)=g(n-1)+d(n)$, where $d(n)$ is the number of positive divisors of $n$. Thus $g(n)=d(1)+d(2)+\cdots+d(n)$ and $f(n)$ is the arithmetic mean of the numbers $d(1),\dots,d(n)$. Therefore, (a) and (b) will follow if we show that each of $d(n+1)>f(n)$ and $d(n+1)<f(n)$ holds infinitely often. But $d(n+1)<f(n)$ holds whenever $n+1$ is prime, and $d(n+1)>f(n)$ holds whenever $d(n+1)>d(1),\dots,d(n)$ (which clearly holds for infinitely many $n$).

27. We first show that every fixed point $x$ of $Q$ is in fact a fixed point of $P\circ P$. Consider the sequence given by $x_0=x$ and $x_{i+1}=P(x_i)$ for $i\ge 0$. Assume $x_k=x_0$. We know that $u-v$ divides $P(u)-P(v)$ for every two distinct integers $u$ and $v$. In particular,
$$d_i=x_{i+1}-x_i \mid P(x_{i+1})-P(x_i)=x_{i+2}-x_{i+1}=d_{i+1}$$
for all $i$, which together with $d_k=d_0$ implies $|d_0|=|d_1|=\cdots=|d_k|$. Suppose that $d_1=d_0=d\ne 0$. Then $d_2=d$ (otherwise $x_3=x_1$ and $x_0$ will never occur in the sequence again). Similarly, $d_3=d$ etc., and hence $x_i=x_0+id\ne x_0$ for all $i$, a contradiction. It follows that $d_1=-d_0$, so $x_2=x_0$ as claimed. Thus we can assume that $Q=P\circ P$.

If every integer $t$ with $P(P(t)) = t$ also satisfies $P(t) = t$, the number of solutions is clearly at most $\deg P = n$. Suppose that $P(t_1) = t_2$, $P(t_2) = t_1$, $P(t_3) = t_4$, and $P(t_4) = t_3$, where $t_1 \neq t_{2,3,4}$ (but not necessarily $t_3 \neq t_4$). Since $t_1 - t_3$ divides $t_2 - t_4$ and vice versa, we conclude that $t_1 - t_3 = \pm(t_2 - t_4)$. Assume that $t_1 - t_3 = t_2 - t_4$, i.e. $t_1 - t_2 = t_3 - t_4 = u \neq 0$. Since the relation $t_1 - t_4 = \pm(t_2 - t_3)$ similarly holds, we obtain $t_1 - t_3 + u = \pm(t_1 - t_3 - u)$ which is impossible. Therefore, we must have $t_1 - t_3 = t_4 - t_2$, which gives us $P(t_1) + t_1 = P(t_3) + t_3 = c$ for some $c$. It follows that all integral solutions $t$ of the equation $P(P(t)) = t$ satisfy $P(t) + t = c$, and hence their number does not exceed $n$.

28. Every prime divisor $p$ of $\frac{x^7-1}{x-1} = x^6 + \cdots + x + 1$ is congruent to 0 or 1 modulo 7. Indeed, if $p \mid x - 1$, then $\frac{x^7-1}{x-1} \equiv 1 + \cdots + 1 \equiv 7 \pmod{p}$, so $p = 7$; otherwise the order of $x$ modulo $p$ is 7 and hence $p \equiv 1 \pmod 7$. Therefore every positive divisor $d$ of $\frac{x^7-1}{x-1}$ satisfies $d \equiv 0$ or $1 \pmod 7$.
Now suppose $(x,y)$ is a solution of the given equation. Since $y - 1$ and $y^4 + y^3 + y^2 + y + 1$ divide $\frac{x^7-1}{x-1} = y^5 - 1$, we have $y \equiv 1$ or 2 and $y^4 + y^3 + y^2 + y + 1 \equiv 0$ or $1 \pmod 7$. However, $y \equiv 1$ or 2 implies that $y^4 + y^3 + y^2 + y + 1 \equiv 5$ or $3 \pmod 7$, which is impossible.

29. All representations of $n$ in the form $ax + by$ $(x, y \in \mathbb{Z})$ are given by $(x,y) = (x_0 + bt, y_0 - at)$, where $x_0, y_0$ are fixed and $t \in \mathbb{Z}$ is arbitrary. The following lemma enables us to determine $w(n)$.
*Lemma*. The equality $w(ax + by) = |x| + |y|$ holds if and only if one of the following conditions holds:
(i) $\frac{a-b}{2} < y \le \frac{a+b}{2}$ and $x \ge y - \frac{a+b}{2}$;
(ii) $-\frac{a-b}{2} \le y \le \frac{a-b}{2}$ and $x \in \mathbb{Z}$;
(iii) $-\frac{a+b}{2} \le y < -\frac{a-b}{2}$ and $x \le y + \frac{a+b}{2}$.
*Proof*. Without loss of generality, assume that $y \ge 0$. We have $w(ax + by) = |x| + y$ if and only if $|x + b| + |y - a| \ge |x| + y$ and $|x - b| + (y + a) \ge |x| + y$, where the latter is obviously true and the former clearly implies $y < a$. Then the former inequality becomes $|x + b| - |x| \ge 2y - a$. We distinguish three cases: if $y \le \frac{a-b}{2}$, then $2y - a \le b$ and the previous inequality always holds; for $\frac{a-b}{2} < y \le \frac{a+b}{2}$, it holds if and only if $x \ge y - \frac{a+b}{2}$; and for $y > \frac{a+b}{2}$, it never holds.
Now let $n = ax + by$ be a local champion with $w(n) = |x| + |y|$. As in the lemma, we distinguish three cases:
(i) $\frac{a-b}{2} < y \le \frac{a+b}{2}$. Then $x + 1 \ge y - \frac{a+b}{2}$ by the lemma, so $w(n+a) = |x+1| + y$ (because $n + a = a(x+1) + by$). Since $w(n+a) \le w(n)$, we must have $x < 0$. Likewise, $w(n-a)$ equals either $|x-1| + y = w(n) + 1$ or $|x+b-1| + a - y$. The condition $w(n-a) \le w(n)$ leads to $x \le y - \frac{a+b-1}{2}$; hence $x = y - [\frac{a+b}{2}]$ and $w(n) = [\frac{a+b}{2}]$. Now $w(n-b) = -x + y - 1 = w(n) - 1$ and $w(n+b) = (x+b) + (a-1-y) = a + b - 1 - [\frac{a+b}{2}] \le w(n)$, so $n$ is a local champion. Conversely, every $n = ax + by$ with $\frac{a-b}{2} < y \le \frac{a+b}{2}$ and $x = y - [\frac{a+b}{2}]$ is

a local champion. Thus we obtain $b-1$ local champions, which are all distinct.

(ii) $|y| \le \frac{a-b}{2}$. Now we conclude from the lemma that $w(n-a) = |x-1|+|y|$ and $w(n+a) = |x+1|+|y|$, and at least one of these two values exceeds $w(n) = |x|+|y|$. Thus $n$ is not a local champion.

(iii) $-\frac{a+b}{2} \le y < -\frac{a-b}{2}$. By taking $x, y$ to $-x, -y$ this case is reduced to case (i), so we again have $b-1$ local champions $n = ax+by$ with $x = y + [\frac{a+b}{2}]$.

It is easy to check that the sets of local champions from cases (i) and (iii) coincide if $a$ and $b$ are both odd (so we have $b-1$ local champions in total), and are otherwise disjoint (then we have $2(b-1)$ local champions).

30. We shall show by induction on $n$ that there exists an arbitrarily large $m$ satisfying $2^m \equiv -m \pmod{n}$. The case $n=1$ is trivial; assume that $n>1$.
Recall that the sequence of powers of 2 modulo $n$ is eventually periodic with the period dividing $\varphi(n)$; thus $2^x \equiv 2^y$ whenever $x \equiv y \pmod{\varphi(n)}$ and $x$ and $y$ are large enough. Let us consider $m$ of the form $m \equiv -2^k \pmod{n\varphi(n)}$. Then the congruence $2^m \equiv -m \pmod{n}$ is equivalent to $2^m \equiv 2^k \pmod{n}$, and this holds whenever $-2^k \equiv m \equiv k \pmod{\varphi(n)}$ and $m, k$ are large enough. But the existence of $m$ and $k$ is guartanteed by the inductive hypothesis for $\varphi(n)$, so the induction is complete.

# A

# Notation and Abbreviations

## A.1 Notation

We assume familiarity with standard elementary notation of set theory, algebra, logic, geometry (including vectors), analysis, number theory (including divisibility and congruences), and combinatorics. We use this notation liberally.
We assume familiarity with the basic elements of the game of chess (the movement of pieces and the coloring of the board).
The following is notation that deserves additional clarification.

- $\mathscr{B}(A,B,C)$, $A-B-C$: indicates the relation of *betweenness*, i.e., that $B$ is between $A$ and $C$ (this automatically means that $A,B,C$ are different collinear points).
- $A = l_1 \cap l_2$: indicates that $A$ is the intersection point of the lines $l_1$ and $l_2$.
- $AB$: line through $A$ and $B$, segment $AB$, length of segment $AB$ (depending on context).
- $[AB$: ray starting in $A$ and containing $B$.
- $(AB$: ray starting in $A$ and containing $B$, but without the point $A$.
- $(AB)$: open interval $AB$, set of points between $A$ and $B$.
- $[AB]$: closed interval $AB$, segment $AB$, $(AB) \cup \{A,B\}$.
- $(AB]$: semiopen interval $AB$, closed at $B$ and open at $A$, $(AB) \cup \{B\}$.
  The same bracket notation is applied to real numbers, e.g., $[a,b) = \{x \mid a \leq x < b\}$.
- $ABC$: plane determined by points $A,B,C$, triangle $ABC$ ($\triangle ABC$) (depending on context).
- $[AB,C$: half-plane consisting of line $AB$ and all points in the plane on the same side of $AB$ as $C$.
- $(AB,C$: $[AB,C$ without the line $AB$.

- $\langle \overrightarrow{a}, \overrightarrow{b} \rangle$, $\overrightarrow{a} \cdot \overrightarrow{b}$: scalar product of $\overrightarrow{a}$ and $\overrightarrow{b}$.
- $a, b, c, \alpha, \beta, \gamma$: the respective sides and angles of triangle $ABC$ (unless otherwise indicated).
- $k(O, r)$: circle $k$ with center $O$ and radius $r$.
- $d(A, p)$: distance from point $A$ to line $p$.
- $S_{A_1A_2\ldots A_n}$, $[A_1A_2\ldots A_n]$: area of $n$-gon $A_1A_2\ldots A_n$ (special case for $n = 3$, $S_{ABC}$: area of $\triangle ABC$).
- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$: the sets of natural, integer, rational, real, complex numbers (respectively).
- $\mathbb{Z}_n$: the ring of residues modulo $n$, $n \in \mathbb{N}$.
- $\mathbb{Z}_p$: the field of residues modulo $p$, $p$ being prime.
- $\mathbb{Z}[x]$, $\mathbb{R}[x]$: the rings of polynomials in $x$ with integer and real coefficients respectively.
- $R^*$: the set of nonzero elements of a ring $R$.
- $R[\alpha], R(\alpha)$, where $\alpha$ is a root of a quadratic polynomial in $R[x]$: $\{a + b\alpha \mid a, b \in R\}$.
- $X_0$: $X \cup \{0\}$ for $X$ such that $0 \notin X$.
- $X^+, X^-, aX + b, aX + bY$: $\{x \mid x \in X, x > 0\}$, $\{x \mid x \in X, x < 0\}$, $\{ax + b \mid x \in X\}$, $\{ax + by \mid x \in X, y \in Y\}$ (respectively) for $X, Y \subseteq \mathbb{R}$, $a, b \in \mathbb{R}$.
- $[x]$, $\lfloor x \rfloor$: the greatest integer smaller than or equal to $x$.
- $\lceil x \rceil$: the smallest integer greater than or equal to $x$.

The following is notation simultaneously used in different concepts (depending on context).

- $|AB|$, $|x|$, $|S|$: the distance between two points $AB$, the absolute value of the number $x$, the number of elements of the set $S$ (respectively).
- $(x, y)$, $(m, n)$, $(a, b)$: (ordered) pair $x$ and $y$, the greatest common divisor of integers $m$ and $n$, the open interval between real numbers $a$ and $b$ (respectively).

## A.2 Abbreviations

We tried to avoid using nonstandard notation and abbreviations as much as possible. However, one nonstandard abbreviation stood out as particularly convenient:

- w.l.o.g.: without loss of generality.

Other abbreviations include:

- RHS: right-hand side (of a given equation).

- LHS: left-hand side (of a given equation).
- QM, AM, GM, HM: the quadratic mean, the arithmetic mean, the geometric mean, the harmonic mean (respectively).
- gcd, lcm: greatest common divisor, least common multiple (respectively).
- i.e.: in other words.
- e.g.: for example.

# B

## Codes of the Countries of Origin

| | | | | | |
|---|---|---|---|---|---|
| ARG | Argentina | HRV | Croatia | POL | Poland |
| ARM | Armenia | HUN | Hungary | POR | Portugal |
| AUS | Australia | IDN | Indonesia | PRI | Puerto Rico |
| AUT | Austria | IND | India | PRK | Korea, North |
| BEL | Belgium | IRL | Ireland | ROU | Romania |
| BGR | Bulgaria | IRN | Iran | RUS | Russia |
| BLR | Belarus | ISL | Iceland | SAF | South Africa |
| BRA | Brazil | ISR | Israel | SCG | Serbia and Montenegro |
| CAN | Canada | ITA | Italy | SGP | Singapore |
| CHN | China | JPN | Japan | SRB | Serbia |
| COL | Colombia | KAZ | Kazakhstan | SVK | Slovakia |
| CUB | Cuba | KOR | Korea, South | SVN | Slovenia |
| CYP | Cyprus | KWT | Kuwait | SWE | Sweden |
| CZE | Czech Republic | LTU | Lithuania | THA | Thailand |
| CZS | Czechoslovakia | LUX | Luxembourg | TUN | Tunisia |
| ESP | Spain | LVA | Latvia | TUR | Turkey |
| EST | Estonia | MAR | Morocco | TWN | Taiwan |
| FIN | Finland | MEX | Mexico | UKR | Ukraine |
| FRA | France | MKD | Macedonia | UNK | United Kingdom |
| FRG | Germany, FR | MNG | Mongolia | USA | United States |
| GDR | Germany, DR | NLD | Netherlands | USS | Soviet Union |
| GEO | Georgia | NOR | Norway | UZB | Uzbekistan |
| GER | Germany | NZL | New Zealand | VNM | Vietnam |
| HEL | Greece | PER | Peru | YUG | Yugoslavia |
| HKG | Hong Kong | PHI | Philippines | | |

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2009-17 Sergei Berlov, RUS – IMO2
2009-19 David Monk, UNK
2009-21 Eugene Bilopitov, UKR
2009-24 Ross Atkins, AUS – IMO1
2009-25 Jorge Tipe, PER
2009-28 József Pelikán, HUN
2009-29 Okan Tekman, TUR

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